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POLYNOMIAL EXPANSIONS OF ANALYTIC FUNCTIONS

By R. P. Boas, Jr.

1. Introduction. An Appell set of polynomials is is defined by the formal power series relations

$$A(t)\,e^{zt} = \sum_{n=0}^{\infty} p_n(z)t^n, \tag{1.1}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n t^n,\ a_0 \neq 0. \tag{1.2}$$

We are concerned with the expansion of analytic functions $F(z)$ in series of Appell polynomials uniformly convergent in regions of the complex plane. The class of functions representable depends on the properties of $A(t)$. If we wish to represent only entire functions of exponential type, it is enough to have $A(t)$ regular at 0 (cf. [6], [8], [2]). On the other hand, simple examples [8] show that when $A(t)$ is only regular in a finite region, it may be necessary to have $F(z)$ entire, or analytic in a strip. When $A(t)$ is entire and of exponential type, Sheffer [8] showed, in particular, that any $F(z)$ regular in a (large enough) neighborhood of 0 is represented in (some) neighborhood of 0: if $A(t)$ is of exponential type 0, every $F(z)$ regular at 0 can be represented in its circle of regularity. Sheffer's results are deduced from the theory of an associated functional equation. Here we shall deduce them more directly from the theory of exponential transforms [3]. Our method allows more precise results to be established when more detailed information about $A(t)$ is available, and moreover allows us to show that general regular functions can be represented in some

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cases when $A(t)$ is entire but not of exponential type, so that Sheffer's sufficient condition is not necessary. Theorem 1 indicates the more precise results obtainable for a special class of ploynomials with $A(t)$ of exponential type; Theorem 5 describes a class of entire functions not of exponential type for which the expansion is still possible. An example is given by $A(t) = e^{t^3}$ if $F(z)$ is regular in an equilateral triangle the centers of whose sides are on the rays joining the origin to the cube roots of 1; then $F(z)$ is represented at least in the inscribed circle of this triangle.

Our method is also applicable to a class of the more general polynomial sets which Sheffer [9] calls sets of type zero and which Steffensen [11] calls poweroids; they may be defined by

$$A(t)\ e^{zH(t)} = \sum_{n=0}^{\infty} p_n(z)\, t^n,\ H(0) = 0,\ H'(0) = 1.$$

We shall not attempt to make an exhaustive study of the expansion properties of these sets, but we give one result to illustrate the possibilities of our method (Theorem 7). Roughly speaking, we show that if $H(t)$ is regular at 0, if $J(t)$, the inverse of $H(t)$, is bounded by h on at least three rays which are the outward normals of a closed polygon; and if $A(t)$ is regular and different from zero in $|z| \leqslant h$, then every function regular in a sufficiently large neighborhood of 0 can be represented near 0.

We consider only representation by uniformly (in fact, absolutely) convergent series, but the method could also be used to discuss the summability of the series.

The method, in outline, is as follows. Let $F(z)$ be regular in a bounded closed convex n-gon $P(n \geqslant 3)$ which we do not necessarily suppose to contain the origin. In the interior of P we can represent $f(z)$ in the form

$$F(z) = \sum_{m=1}^{n} \int_{C_m} f_m(t)\, e^{\alpha_m tz}\, dt, \tag{1.3}$$

where the α_m's are complex numbers (all different) of absolute value 1; $f_m(t)$ are entire functions satisfying inequalities of the form

$$|f_m(u+iv)| \leqslant h_m e^{c_m u + b_m |v|}, \tag{1.4}$$

where h_m, c_m, b_m, are certain constants; and C_m are curves extending from 0 to ∞ and close to the positive real axis. From (1.1) we have

$$e^{\alpha_m zt} = \sum_{j=0}^{\infty} p_j(z)(\alpha_m t)^j / A(\alpha_m t). \tag{1.5}$$

If we can substitute (1.5) into (1.3) and integrate term by term, we obtain the desired expansion,

$$F(z) = \sum_{j=0}^{\infty} p_j(z) \sum_{m=1}^{n} \int_{C_m} f_m(t)(\alpha_m t)^j \{A(\alpha_m t)\}^{-1} dt. \tag{1.6}$$

A simple sufficient condition which will make the formal process valid is the convergence of the n series

$$\sum_{j=0}^{\infty} |p_j(z)| \int_{C_m} |f_m(t)|\, |t^j|\, |A(\alpha_m t)|^{-1} |dt|. \tag{1.7}$$

Since sharp estimates for $|p_j(z)|$ are available, the convergence of (1.7) depends on the amount of information available about the magnitude of $1/|A(t)|$.

2. **Representation by a sum of exponential transforms.** We begin by establishing the representation (1.3), which was introduced in a slightly less general form in [3]. Let the polygon P be the intersection of the n half planes $\mathcal{R}(te^{-i\varphi_k}) \geqslant c_k$, let its sides be L_k, and let z be inside P. Then if $F(z)$ is regular in P,

$$F(z) = \sum_{k=1}^{n} (2\pi i)^{-1} \int_{L_k} (t-z)^{-1} F(t)\, dt. \tag{2.1}$$

We put $\alpha_k = e^{-i\varphi_k}$, so that for t on L_k we have $\mathcal{R}(t\alpha_k) = -c_k$. Then for z inside P and t on L_k, we have

$$\alpha_k \int_0^\infty e^{-\alpha_k(t-z)u} du = (t-z)^{-1}, \tag{2.2}$$

since $\mathcal{R}(\alpha_k(t-z)) > 0$. Moreover, for fixed z inside P the integral is absolutely and uniformly convergent (in t) for t on L_k. Thus we have

$$(2\pi i)^{-1} \int_{L_k} (t-z)^{-1} F(t)\, dt$$

$$= \int_0^\infty e^{\alpha_k z u} du \int_{L_k} (2\pi i)^{-1} \alpha_k F(t)\, e^{-\alpha_k t u} dt = \int_0^\infty e^{\alpha_k z u} f_k(u)\, du,$$

where

$$f_k(u) = (2\pi i)^{-1} \alpha_k \int_{L_k} F(t)^{-\alpha_k t u} dt,$$

so that

$$|f_k(u+iv)| \leqslant h_k(e^{c_k u + b_k |v|}), \tag{2.3}$$

where b_k is the largest value of $|\mathcal{R}(-i\alpha_k t)|$, t on L_k. This gives us the representation (1.3) with C_m the real axis; and the real axis can be replaced by any other curve C_m starting at 0 and ending at ∞ which meets the real axis at a sequence of points $t_j \to \infty$, provided that we interpret

$\int_{Cm}$ as $\lim_{j\to\infty} \int_0^{t_j}$.

3. **Generating function with regularly distributed zeros.** We consider the special case when the zeros of $A(t)$ are "measurable" in the sense of Pfluger. This means that for every two points of continuity of a monotone increasing function $\mathcal{N}(\phi)$ the number of zeros in $|z| \leqslant r$, $\phi' \leqslant \arg z < \phi''$ is $[\mathcal{N}(\phi'') - \mathcal{N}(\phi')]\, r + o(r)$. Concerning such functions Pfluger [7] has given a result which contains the following statement. For any ϕ interior to an angle in which $A(t)$ has no zeros,

$$r^{-1} \log |A(re^{i\phi})| = h(\phi) + o(1), \tag{3.1}$$

where $h(\phi)$ is the indicator function of $A(t)$. Then we obtain a majorant series for (1.7) by using (3.1) and the relation [8, 10]

$$\limsup_{n\to\infty} |n!\,p_n(z)|^{1/n} = \max|h(\phi)+\rho\cos(\theta+\phi)| \equiv k(z), \tag{3.2}$$

say, where $z=\rho e^{i\theta}$. We may establish (3.2) by noting that $A(t)e^{zt}$, as a function of t, is of order 1 and type $\max_\phi|h(\phi)+\rho\cos(\theta+\phi)|$; on the other hand, the usual formula for calculating the type from the coefficients states that the type is $\limsup_{n\to\infty}|n!\,p_n(z)|^{1/n}$. Thus if $A(t)$ has no zeros in an angle around each line $\arg t = \pi-\phi_m$, a majorant for (1.7) is

$$\sum_{j=0}^{\infty}\{k(z)\}^j\{h(\pi-\phi_m)-c_m\}^{-j-1}$$

We may thus state the following theorem.

THEOREM 1. *Let $A(t)$ be an entire function with zeros measurable in the sense of Pfluger. Let $F(z)$ be regular in the closed bounded convex n-gon P defined by $\mathcal{R}(ze^{-i\phi_m}) \geqslant c_m$ $(m=1,2,\ldots,n)$. For each m, let $A(t)$ have no zeros in some angle containing $\arg t=\pi-\phi_m$. Then if the region defined by $k(z)<h(\pi-\phi_m)-c$, $m=1,2,\ldots,n$, is a nonempty subset of P (where $h(\phi)$ is the indicator function of $A(t)$ and $k(z)$ is defined by (3.2)), $F(z)$ is repsesented in every closed subregion by a uniformly convergent series of Appell polynomials generated by $A(t)$.*

A general bounded convex region may be treated by approximating it by convex polygons.

As the simplest example let $A(t)=e^{at}$, so that the polynomials are simply

$(z+a)^n/n!$, $k(z)=|z+a|$, $h(\phi)=|a|\cos(\alpha+\phi)$, $\alpha=\arg a$.
The region of convergence given by Theorem 1 is

$$|z+a| < -|a|\cos(\phi_m-\alpha)-c_m,\quad m=1,2,\ldots,n,$$

which is just the largest circle with center at $z=-a$ contained in P. Thus in this trivial case Theorem 1 actually gives the largest possible region of convergence.

For a less trivial example let $A(t)$ be an even function with real zeros having density $2D$. Then $h(\phi)=$

$\pi D|\sin\varphi|$. The function $k(z)$ is the distance from $|z|$ to the more distant of the points $\pm i\pi D$, i.e.

$$k(z) = \{x^2+(y+\pi D\,\mathrm{sgn}\,y)^2\}^{1/2}.$$

The region of convergence given by Theorem 1 is thus

$$x^2+(y+\pi D\,\mathrm{sgn}\,y)^2 < \{\pi D|\sin\phi|-C_m\}^2, \qquad (3.3)$$

and will be nonempty if $c_m < \pi D(|\sin\phi_m|-1)$. In this case the polygon P must contain the point o. If $D=0$, (3.3) is just the largest circle with center at o contained in P; this agrees with the general result [8] for $A(t)$ of zero exponential type (cf. § 6 below).

4. **Generating function of order less than 1.** The theorem of this section is a special case of the theorem of Sheffer which will be proved in §6, but its proof is considerably simpler.

THEOREM 2. *Let $A(t)$ be an entire function of order less than 1. If $F(z)$ is regular at $z=0$, it is represented in its largest circle of regularity about $z=0$ by a series of the Appell polynomials generated by $A(t)$.*

Let $F(z)$ be regular in a bounded closed convex polygon P containing o in its interior. It is well known that $|A(t)| \geqslant e^{-o(t)}$ if t is outside a set of circles whose total length is finite. In the representation (1.3) let C_m be the real axis with indentations (of finite total length) which avoid the exceptional circles for $A(\alpha_m t)$. Then (1.6) is valid if

$$\sum_{j=0}^{\infty}\frac{\{k(z)\}^j}{j!}\int_{C_m}|F_m(t)|\,|t|^j e^{\varepsilon|t|}\,|dt| < \infty, \qquad (4.1)$$

for some (small) $\varepsilon > 0$. Moreover, $h(\phi)=0$ and so $k(z)=|z|$. If s is arc length on C_m and C_m is defined by $t=t(s)$, (4.1) thus amounts to

$$\sum_{j=0}^{\infty}\frac{|z|^j}{j!}\int_0^{\infty}|F_m(t(s))|\,|t(s)|^j e^{\varepsilon|t(s)|}ds < \infty,$$

or, by (1.4), since $s-a \leqslant |t(s)| \leqslant s+a$, where a is a constant,

$$\sum_{j=0}^{\infty} \frac{|z|^j}{j!} \int_0^{\infty} e^{c_m(s-a)}(s+a)^j e^{\varepsilon(s+a)} ds < \infty,$$

provided that all $c_m < 0$, and this series in fact converges if $\varepsilon < \gamma = \min(-c_m)$ and $|z| < \gamma - \varepsilon$. Since ε can be arbitrarily small, (4.1) is true for all z in the largest circle with center at the origin, contained in P.

5. **A lemma on entire functions.** In order to discuss the case where $A(t)$ is a general entire function of exponential type, we need the following lemmas. As M.L. Cartwright has pointed out to me, they can be derived in a simple way from a lemma of Cartwright [4] and V. Bernstein [1]; however, we give a direct proof here, since it leads to better estimates for the constants involved in the lemmas.

LEMMA 1. *Let $f(z)$ be an entire function of order 1 and zero type, with $f(0) = 1$. There exists a function $\varepsilon(r)$ such that $\varepsilon(r) = o(r)$ as $r \to \infty$, and a set of circles such that the sum of the diameters of the circles inside $|z| < r$ is at most $\varepsilon(r)$, with the property that for z outside the circles we have*

$$\liminf_{|z|\to\infty} |z|^{-1} \log |f(z)| \geqslant 0.$$

LEMMA 2. *Let $f(z)$ be an entire function of order 1 and mean type, with $f(0) = 1$. For every positive b there exist a finite $c(b)$ and a set of circles such that the sum of the diameters of the circles inside $|z| < r$ is at most br, with the property that* $\log |f(z)| \geqslant c|z|$ *for z outside the circles.*

Let the zeros of $f(z)$ be $z_n = r_n e^{i\theta_n}$, $r_n > 0$, with the r_n arranged in order of increasing magnitude. As usual, let $n(r)$ denote the number of r_n not exceeding r. There are two cases, according as $\sum 1/r_n$ converges or diverges. In the first case $f(z)$ is necessarily of zero type, and we have

$$f(z) = \prod_{n=1}^{\infty} (1 - z/z_n). \qquad (5.1)$$

Let $r = |z|$ and $N = [\log_2 r]$ (where $\log_2$ means logarithm to the base 2), so that $2^N \leqslant r < 2^{N+1}$. We write

$f(z) = P_1 . P_2$, where P_1 contains the terms of the product in (5.1) for which $r_n \leqslant 2^{N+2}$, P_2 the rest. Since $2^{N+2} > 2r$ we have $r_n > 2r$ in P_2, and so $|z/z_n| < \frac{1}{2}$, $\log|1-z/z_n| \geqslant -2|z/z_n|$,

$$\log|P_2| \geqslant -2r \sum_{r_n \geqslant 2r} r_n^{-1} = -\varepsilon_1(r), \qquad (5.2)$$

say, where $\varepsilon_1(r) = o(r)$. To estimate P_1, we use the Boutroux-Cartan lemma [12, p. 10]. This states that, for arbitrary positive H,

$$\left|\prod_{\nu=1}^{n} (z-z_\nu)\right| > H^n$$

if z is outside n circles the sum of whose radii is $2eH$. Applied to P_1, it says that

$$\log|P_1| \geqslant n(2^{N+2}) \log H + \log \prod_{r_n \leqslant 2N+2} r_n^{-1}$$

if z is outside certain circles the sum of whose radii is $2eH$. In P_1 we have

$$r_n \leqslant 2^{N+2} \leqslant 4r,\ r_n^{-1} \geqslant r^{-1},\ \Pi r_n^{-1} \geqslant \Pi(\tfrac{1}{4}r^{-1}) = (4r)^{-n(2^{N+2})}$$

Thus outside circles the sum of whose radii is $2eH$ we have

$$\log|P_1| \geqslant n(2^{N+2})(\log H - \log 4r)$$
$$\geqslant n(2^{N+2}) \log (H/2^{N+3}). \qquad (5.3)$$

We now choose $H = H(N)$ so that $H(N) = o(2^N)$ as $N \to \infty$ and also so that $n(2^{N+2}) \log\{H(N)/2^{N+3}\} = o(2^N)$; this choice is possible because $n(2^{N+2}) = o(2^{N+2})$ and so the second condition will be fulfilled if $H(N)/2^N \to 0$ slowly enough. If we write $n(2^{N+2}) \log (H/2^{N+3}) = \varepsilon_2(r) = o(r)$, and $\varepsilon(r) = -\varepsilon_1(r) + \varepsilon_2(r)$, we have, from (5.2) and (5.3), $\log|f(z)| \geqslant \varepsilon(r)$ for $2^N \leqslant r < 2^{N+1}$, provided that z is outside circles the sum of whose radii is $\delta(2^N) = 4e\pi H(N) = o(2^N)$. Then the total sum of the radii of the exceptional circles inside $|z| < r$ is at most $\delta(2) + \delta(2^2) + \dots + \delta(2^N) = o(2^{N+1}) = o(r)$. This completes the proof of Lemma 1 in the first case.

In the second case we have

$$f(z) = e^{az} \prod_{n=1}^{\infty} (1 - z/z_n)\, e^{z/z_n}.$$

As before, we let $r=|z|$ and $N=[\log_2 r]$, and write $f(z)=P_1 . P_2$, where P_2 contains the terms of the product for which $r_n > 2^{N+2}$. Then we have, in P_2,

$$\log|(1-z/z_n)\, e^{z/z_n}| \geqslant -2|z/z_n|^2.$$

Hence

$$\log|P_2| \geqslant -2r^2 \sum_{r_n \geqslant 2r} r_n^{-2} = -2r^2 \int_{2r}^{\infty} t^{-2} du(t)$$

$$= \tfrac{1}{6} n(2r) - 4r^2 \int_{2r}^{\infty} t^{-3}\, n(t)\; dt.$$

If $f(z)$ is of zero type, we have $n(t)=o(t)$, the last integral is $o(r)$, and so

$$\log|P_2| \geqslant o(r). \tag{5.5}$$

If $f(z)$ is of positive type τ, and we put $N(t)=\int_0^t u^{-1} n(u)\, du$, we have $\limsup_{r\to\infty} r^{-1} N(r) \leqslant \tau$. Then

$$\int_{2r}^{\infty} t^{-3} n(t)\, dt = \int_{2r}^{\infty} t^{-2} dN(t) = -\tfrac{1}{4} r^{-2} N(2r) + 2\int_{2r}^{\infty} t^{-3} N(t)\, dt.$$

Using this in (5.4), we have, for large r,

$$\log|P_2| \geqslant -8r^2 \int_{2r}^{\infty} t^{-3} N(t)\, dt \geqslant -4r\tau. \tag{5.6}$$

Next we write $\log|P_1|$ in the form

$$\log|P_1| = \mathscr{R}\{\alpha z + \Sigma\, z/z_n\} + \log|\Pi(1-z/z_n)| = L_1 + L_2,$$

say, where Σ and Π are over $r_n \leqslant 2^{N+2}$. We consider L_1 first. If $f(z)$ is of type zero, we have $\Sigma_{n=1}^{\infty} z_n^{-1} = \alpha$ and so $L_1 = o(r)$. If $f(z)$ is of positive type, the sums $\alpha + \Sigma_{r_n \leqslant r} z_n^{-1}$ are bounded [5, p. 375]; let σ be a bound for them. Then

$$L_1 \geqslant -\sigma r. \tag{5.7}$$

Now L_2 has the same form as $\log|P_1|$ in the first case of the lemmas and so (5.3) applies:

$$L_2 \geqslant n(2^{N+2}) \log(H/2^{N+3}), \tag{5.8}$$

if $2^N \leqslant |z| < 2^{N+1}$ and z is outside certain circles the sum of whose radii is $2eH$. If $f(z)$ is of type zero, we still

have $n(t)=o(t)$ and so we get $L_2 \geqslant o(|z|)$ if z is outside circles whose total length inside $|z|<r$ is $o(r)$; hence $\log |f(z)| \geqslant o(r)$ outside the same circles.

If $f(z)$ is of positive type, we choose $H(N) = 2^N b/(8e)$, where b is any positive number. Then by (5.8), $L_2 \geqslant n(2^{N+2}) \log (b/e) \geqslant n(2r) \log (b/e)$ if $2^N \leqslant |z| < 2^{N+1}$ and z is outside circles the sum of whose radii is $2^{N-2}b$. Then the sum of the diameters of the excluded circles inside $|z|<r$ is at most $2^N b \leqslant br$. Using (5.6) and (5.7), we have the result required.

6. Generating function of exponential type.

THEOREM 3 [8]. *Let $A(t)$ be an entire function of order* 1 *and minimum type. If $F(z)$ is regular at $z=0$, it is represented in its largest circle of regularity about $z=0$ by a series of the Appell polynomials generated by $A(t)$.*

The proof depends on Lemma 1 and is just like that of Theorem 2 except that now $s-o(s) \leqslant t(s) \leqslant s+o(s)$. This demands only trivial changes in the verification of (4.1).

THEOREM 4 [8]. *Let $A(t)$ be an entire function of order* 1 *and mean type. If $F(z)$ is regular in a sufficiently large neighborhood of* 0, *it is represented in some neighborhood of* 0 *by a series of the Appell polynomials generated by $A(t)$.*

Now we use Lemma 2 and follow the argument of §4 as closely as possible. We now require

$$\sum_{j=0}^{\infty} \frac{\{k(z)\}^j}{j!} \int_{C_m} |F_m(t)|\, |t|^j e^{c|t|}\, |dt| < \infty, \qquad (6.1)$$

where $c=c(b)$ is the constant of Lemma 2 and $(1-b)s \leqslant |t(s)| \leqslant (1+b)s$. Thus if all $c_m<0$, (6.1) amounts to

$$\sum_{j=0}^{\infty} \frac{\{k(z)\}^j}{j!} \int_0^{\infty} e^{(b_m+c_m)(1-b)s}((1+b)s)^j e^{c(1+b)s} ds < \infty,$$

where b_m is defined in (2.3), or

$$\sum_{j=0}^{\infty} \{k(z)\}^j (1+b)^j / \{|c+b_m|(1-b)-c(1+b)\}^j, \quad (6.2)$$

provided that $|c_m+b_m|(1-b) > c(1+b)$, which is always true if $c_m < 0$ and $|c_m|$ is large enough: and by taking $|c_m|$ still larger, if necessary (i.e. by taking a large enough polygon P containing 0), we can make (6.2) converge.

7. **A wider class of generating functions.** Only when $A(t)$ is an entire function of exponential type do we have an estimate of the form (3.2) for $P_n(z)$. However, in estimating (1.7) we can allow $P_n(z)$ to be larger if $1/A(t)$ is sufficiently small in certain directions. The following theorem indicates some of the possibilities in this direction.

THEOREM 5. *Let $A(t)$ be an entire function, $M(r) = \max|A(t)|$ for $|t| = r$. Suppose that $rM'(r)/M(r)$ increases, that $\omega(r)$ is its inverse, and that*

$$\Sigma e^{(b-c)\omega\{n/(1+c)\}} \tag{1.7}$$

converges for some (small) b and some (large) c. Suppose finally that there are at least three rays which are the outward normals to the sides of a closed convex polygon and on which $|A(t)| \geqslant \lambda M(t)$, where λ is a positive constant. Then every function regular in a sufficiently large neighborhood of 0 is represented in $|z| < b$ by a series of the Appell polynomials generated by $A(t)$.

For example, the hypotheses are satisfied by $A(t) = e^{t^n}$, where n is an integer > 2. In this case the series (7.6) converges if $b < c$, and it is easy to see that every function regular in a regular n-gon with center at 0 and sides perpendicular to the rays joining 0 to the nth roots of 1 is represented in the inscribed circle of the n-gon.

PROOF. We have

$$p_n(z) = (2\pi i)^{-1}\int_{|t|=r} t^{-n-1} e^{zt} A(t)\, dt,$$

where r is at our disposal. Then we have

$$|p_n(z)| \leqslant \min_r e^{r|z|} M(r) r^{-n}. \tag{7.2}$$

Let $|A(\alpha_m t)| \geqslant \lambda M(|t|)$ for at least three α_m's such that the polygon $\mathcal{R}(-\alpha_m z) > c_m$ is closed. Then a majorant for (1.7) is

$$\lambda^{-1}\sum_{n=0}^{\infty} \min_r \, e^{r|z|} M(r)\, r^{-n} \int_0^\infty e^{c_m t}\, t^n \left\{ M(t) \right\}^{-1} dt.$$

For convenience, write $|z| = b$ and $c_m = -c \;-\varepsilon$, $c > 0$, $\varepsilon > 0$. Then the convergence of (1.7) is implied by that of

$$\sum_{n=0}^{\infty} \min_r \left(e^{br} M(r)\, r^{-n}\right). \max_t \left(e^{-ct} \left\{ M(t) \right\}^{-1} t^n\right). \int_0^\infty e^{-\varepsilon t}\, dt$$

$$= \epsilon^{-1} \sum_{n=0}^{\infty} \frac{\min\left(e^{br}\; M(r)\; r^{-n}\right)}{\min\left(e^{cr}\; M(r)\; r^{-n}\right)}. \tag{7.3}$$

Suppose that the minimum in the demoninator in (7.3) occurs at $r = r_n$; then the fraction does not exceed

$$e^{br_n} M(r_n)\, r_n^{-n} \;/\; e^{cr_n} M(r_n)\, r_n^{-n} \;=\; e^{(b-c)r_n}.$$

To estimate r_n, we observe that

$$cr_n + r_n\, M'(r_n) \,/\, M(r_n) = n,$$

and so $(1+c)\, r_n\, M'(r_n) \,/\, M(r_n) > n$,

$$r_n > \omega[n \,/\, (1+c)].$$

By hypothesis, $\Sigma e^{(b-c)r_n}$ converges if b is small and c is large. Hence (1.7) converges if $|z|$ is small and all $-c_m$ are large, i.e. if $F(z)$ is regular in a sufficiently large neighborhood of 0.

It is clear that considerably sharper results are obtainable if more careful estimates are made in the proof of Theorem 5, particularly for special choices of $A(t)$. However, even in its present form Theorem 5 adequately indicates that the class of generating functions which furnish Appell sets suitable for expanding general regular functions is considerably larger than the class of entire functions of exponential type.

8. **Another class of polynomials.** Let $A(t)$ be given by (1.2), and let $H(t)$ be regular at 0, with $H(0) = 0$, $H'(0) = 1$; let $J(t)$ be the inverse of $H(t)$. Then the polynomials in question (Sheffer's [9] sets of type zero; Steffensen's [11] poweroids) are defined by

$$A(t)\, e^{zH(t)} = \sum_{n=0}^{\infty} p_n(z)\, t^n \tag{8.1}$$

or alternatively by

$$A[J(t)]e^{zt} = \sum_{n=0}^{\infty} p_n(z)\, [J(t)]^n. \tag{8.2}$$

If we use (8.2) we see that the series corresponding to (1.7), whose convergence will ensure the uniform convergence of an expansion in terms of the $p_n(z)$ of a function regular in the polygon P, is

$$\sum_{n=0}^{\infty} |p_n(z)| \int_{C_m} |f_m(t)|\, |J(\alpha_m t)|^n |A[J(\alpha_m t)]|^{-1} |dt|, \tag{8.3}$$

where we now have to assume explicitly that $J(\alpha_m t)$ is defined on all the curves C_m. Since we are not seeking great generality, but only some results to illustrate the possibilities of our method, we shall suppose that C_m can be taken to be the positive real axis. If we use (1.4) and an estimate for $|p_n(z)|$ analogous to (7.2), we are thus led to consider the series

$$\sum_{n=0}^{\infty} \min_{0<t<R} \{ t^{-1} A(t)\, e^{|zH(t)|} \} \int_0^{\infty} e^{c_m t} |J(\alpha_m t)|^n |A[J(\alpha_m t)]|^{-1} dt, \tag{8.4}$$

where $A(t)$ and $H(t)$ are regular in $|t| < R$ ($\leqslant \infty$). Suppose that $|J(\alpha_m t)| \leqslant h$ for each m, that $A(t)$ is regular and never zero in $|t| \leqslant h$, and that $S < R$. Then the convergence of (8.4) is implied by that of

$$\sum_{n=0}^{\infty} S^{-n} h^n \int_0^{\infty} e^{c_m t}\, dt,$$

which converges if $c_m < 0$ and $S > h$. Hence we have the following theorem.

THEOREM 7. *Let $F(z)$ be regular in the polygon*

$$\mathscr{R}(-\alpha_m z) \geqslant c_m,\ c_m < 0,\ m = 1, 2, \ldots, n.$$

If $|J(\alpha_m t)| \leqslant h$, $0 \leqslant t < \infty$, for each m, if $A(t)$ and $H(t)$ are regular in $|t| < R$ $(R > h)$ and if $A(t) \neq 0$ in $|t| \leqslant h$, then $F(z)$

is represented in the interior of P by a series of the polynomials defined by (8.1).

For example, we might have $J(t) = te^{t^2}$, $\alpha_1 = i$, $\alpha_2 = e^{2\pi i/3}$, $\alpha_3 = -i$, $\alpha_4 = e^{i\pi/3}$, $h = e^{-\frac{1}{2}}$; it is enough to have $A(t)$ regular in $|t| \leqslant 1$ and not zero in $|t| \leqslant e^{-\frac{1}{2}}$.

For another example, let $J(t) = t(1+t)^{-1}$, $H(t) = t(1-t)^{-1}$, ; $\alpha_1 = 1$, $\alpha_2 = e^{2\pi i/3}$, $\alpha_3 = e^{-2\pi i/3}$, $h = 1$. It is enough to have $A(t)$ regular and not zero in $|t| \leqslant 1$. For Laguerre polynomials $A(t) = (1-t)^{-1}$ and Theorem 7 just fails to apply; in fact, it is known that not all functions regular in a neighborhood of 0 can be represented by series of Laguerre polynomials.

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DIFFERENTIAL EQUATIONS OF INFINITE ORDER

By R. P. Boas, Jr.

1. Introduction. We consider the equation

$$\sum_{k=0}^{\infty} a_k F^{(k)}(z) = G(z), a_0 \neq 0, \qquad (1.1)$$

where the a_k are constants and $G(z)$ is regular in a region which will be specified later. We shall use the formalism of exponential transforms [1] [2] to show the existence of a solution of (1.1) under suitable conditions on the a_k and on $G(z)$. Let

$$A(t) = \sum_{k=0}^{\infty} a_k t^k. \qquad (1.2)$$

If $A(t)$ is an entire function of exponential type, Sheffer[5] showed that (1.1) has a solution $F(z)$ regular in some neighborhood of 0 provided that $G(z)$ is regular in a large enough neighborhood of 0; if the type of $A(t)$ is zero, $F(z)$ exists in the largest circle of regularity of $G(z)$ with center at 0 (see also Muggli [3]). We shall give new proofs of these results, the second in the sharper form that $F(z)$ exists in any convex region containing 0 in the domain of regularity of $G(z)$; we also have sharper results when $A(t)$ belongs to special classes of functions of exponential type. Moreover, we obtain regular solutions of (1.1) with $G(z)$ merely regular in a large enough neighbourhood of 0 even when $A(t)$ is an entire function which is not of exponential type but is suitably restricted in other ways.

The results of this paper parallel the results of the preceding paper [2] on polynomial expansions. we refer to that paper for details omitted here.

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The idea of the discussion is as follows. If $F(z)$ is regular in a closed bounded convex polygon P, it has the representation [1] [2]

$$F(z) = \sum_{m=1}^{n} \int_{C_m} e^{\alpha_m z t} f_m(t)\, dt, \tag{1.3}$$

where α_m are (different) complex numbers of absolute value 1, C_m is a curve close to the positive real axis, and $f_m(t)$ are entire functions such that

$$|f_m(t)| \leqslant h_m e^{c_m u + b_m |v|}, \quad t = u + iv \tag{1.4}$$

(h_m, c_m, b_m are constants depending on P). Now we have

$$F^{(k)}(z) = \sum_{m=1}^{n} \alpha_m{}^k \int_{C_m} t^k e^{\alpha_m z t} f_m(t)\, dt, \tag{1.5}$$

and, at least formally,

$$\sum_{k=0}^{\infty} a_k F^{(k)}(z) = \sum_{m=1}^{n} \int_{C_m} f_m(t) e^{\alpha_m z t} \sum_{k=0}^{\infty} (\alpha_m t)^k dt \tag{1.6}$$

$$= \sum_{m=1}^{n} \int_{C_m} f_m(t) A(\alpha_m t) e^{\alpha_m z t} dt;$$

moreover,

$$G(z) = \sum_{m=1}^{n} \int_{C_m} e^{\alpha_m z t} g_m(t)\, dt.$$

Consequently, (1.1) is formally satisfied by taking

$$f_m(t) = g_m(t)/A(\alpha_m t), \qquad m = 1, 2, \ldots, n.$$

If $A(\alpha_m t)$ has no zeros on C_m, this choice will actually give us a solution of (1.1) provided that we can pass from (1.6) to (1.5). A simple condition permitting this is that for each m

$$\sum_{k=0}^{\infty} |a_k| \int_{C_m} |t|^k |e^{\alpha_m t z}| |g_m(t)| |A(\alpha_m t)|^{-1} |dt| < \infty, \tag{1.7}$$

and the remainder of this paper is concerned with verifying (1.7) in various cases.

2. **Special generating functions.** Consider first the case when C_m is the positive real axis. Then (1.7) is equivalent to

$$\sum_{k=0}^{\infty} |a_k| \int_0^{\infty} u^k e^{u\mathbf{R}(\alpha_m z)} e^{c_m u} |A(\alpha_m u)|^{-1} du. \qquad (2.1)$$

THEOREM 1. *Let $A(t)$ be an entire function of exponential type τ, whose zeros are measurable in the sense of Pfluger.* Let $G(z)$ be regular in the closed bounded convex region P defined by $\mathcal{R}(ze^{-i\varphi_m}) \geqslant c_m$, $m=1, 2, \ldots, n$. For each m, let $A(t)$ have no zeros in some angle containing $\arg t = \pi - \varphi_m$. Then if $-c_m$ is sufficiently large, there is a solution $F(z)$ of (1.1) which is regular at least in the region defined by*

$$\mathcal{R}(ze^{-i\varphi_m}) > c_m - h(\pi - \varphi_m) + \tau,$$

where $h(\varphi)$ is the indicator function of $A(t)$.

Let $\alpha_m = -e^{-i\varphi_m}$. According to Pfluger, we have $\log |A(\alpha_m u)| = uh(\pi - \varphi_m) + o(u)$. Furthermore, lim sup $|k!\, a_k|^{1/k} = \tau$, the type of $A(t)$. Then the convergence of (2.1) is equivalent to that of

$$\sum_{k=0}^{\infty} (\tau^k/k!) \int_0^{\infty} u^k e^{u[\mathbf{R}(\alpha_m z) - h(\pi-\varphi_m) + c_m]} du,$$

or of

$$\sum_{k=0}^{\infty} \tau^k / \{ \mathcal{R}(ze^{-i\varphi_m}) + h(\pi - \varphi_m) - c_m \}^k. \qquad (2.2)$$

If z is restricted to the region Q defined by $\mathcal{R}(ze^{-i\varphi_m}) > b_m$, where $b_m > c_m - h(\pi - \varphi_m) + \tau$, series (2.2) converges. The region Q will not be empty if $-c_m$ is large enough; and if $h(\pi - \varphi_m) = \tau$, so that $A(t)$ grows at its maximum rate in the directions $\pi - \varphi_m$, Q is any subpolygon of P (with sides parallel to those of P).

* Pfluger [4]. For every two points of continuity of a monotone increasing function $N(\varphi)$, the number of zeros in $|z| \leqslant r$, $\varphi' \leqslant \arg z < \varphi''$, is $[N(\varphi'') - N(\varphi')]r + o(r)$.

Before considering general functions of exponential type, we consider a class of functions $A(t)$ which are entire but not of exponential type. We shall not try to obtain sharp results, but our next theorem will at least show that (1.1) possesses regular solutions for an extensive class of functions which are not of exponential type.

THEOREM 2. *Let $A(t)$ be an entire function, $M(r) = \max |A(t)|$ for $|t| = r$. Suppose that $rM'(r)/M(r)$ increases that $\omega(r)$ is its inverse, and that*

$$\sum e^{-c\omega[n/(1+c)]} \tag{2.3}$$

converges for sufficiently large c. Suppose finally that there are at least three rays which are the outward normals to the sides of a closed convex polygon and on which $|A(t)| \geqslant \lambda M(t)$, where λ is a positive constant. Then if $G(z)$ is regular in a sufficiently large neighborhood of 0, there is a solution of (1.1) regular in some neighborhood of 0.

We have $|a_k| \leqslant \min_r M(r)r^{-k}$. Let $|A(\alpha_m t)| \geqslant \lambda M(t)$ for at least three different α_m's, such that the polygon $\mathcal{R}(-\alpha_m z) > c_m$ is closed. Then a majorant for (2.1) is

$$\lambda^{-1} \sum_{k=0}^{\infty} \min_r \{ M(r) r^{-k} \} . \int_0^{\infty} u^k e^{u\,[\mathrm{R}(\alpha_m z)+c_m]} \{ M(u) \}^{-1} du. \tag{2.4}$$

Let $\mathcal{R}(\alpha_m z)+c_m = -c-\varepsilon \leqslant c > 0, \varepsilon > 0$. Then (2.1) converges if

$$\sum_{k=0}^{\infty} \min_r \{ M(r)\, r^{-k} \} . \max_u \{ e^{-cu} u^k\, M(u)^{-1} \} . \int_0^{\infty} e^{-\varepsilon u}\, du$$

converges, and thus if

$$\sum_{k=0}^{\infty} \min \{ M(r)\, r^{-k} \} / \min \{ e^{cr}\, M(r)\, r^{-k} \}$$

converges. Let the minimum in the denominator occur at $r = r_k$; then the fraction does not exceed

$$M(r_k)\, r_k^{-k} / \{ e^{cr_k} M(r_k)\, r_k^{-k} \} = e^{-cr_k}.$$

Since $cr_k + r_k\, M'(r_k)/M(r_k) = k$, we certainly have $r_k > \omega(k/(1+c))$, and so by (2.3) Σe^{-cr_k} converges if c is sufficiently large, i.e. if $\mathcal{R}(e^{-i\varphi_m} z) > c_m + b$ with a sufficiently large b. The region defined by these inequalities will be nonempty if $-c_m$ is large enough.

For example, if $A(t) = e^{t^n}$, where n is an interger greater than 2, the series (2.3) converges if $c > 0$, and if the sides of the polygon P are perpendicular to the directions of maximum growth of $A(t)$, there is a solution regular in P.

3. **Generating function of exponential type.** We shall show how our method leads to Sheffer's theorems for general functions of exponential type. In estimating (1.7), we suppose that C_m is a curve which avoids the zeros of $A(\alpha_m u)$. It is known (see [2]) that C_m can be chosen so that if s denotes arc length on C_m and C_m is defined by $t = t(s)$, we have $(1-b)\, s \leqslant t(s) \leqslant (1+b)\, s$, where $0 < b < 1$, and $|A(\alpha_m t)| \geqslant e^{c|t|}$, where c depends on b; if $A(t)$ is of zero exponential type, both b and c may be taken arbitrarily small. Then a majorant for (1.7) is (if all $c_m < 0$)

$$\sum_{k=0}^{\infty} (\tau^k/k!) \int_0^{\infty} [(1+b)s]^k e^{(b_m + c_m)\,(1-b)\, s}\, e^{c\,(1+b)\, s}\, e^{|z|\,(1+b)\, s}\, ds,$$

where b_m is defined by (1.4), or

$$\sum_{k=0}^{\infty} [\tau(1+b)]^k / \{ |b_m + c_m|\, (1-b) - (c + |z|)\, (1+b) \},$$

which converges provided $|c_m|$ is large enough. In particular, when the type of $A(t)$ is zero, τ, b and c are all arbitrarily small and b_m does not appear, so that the convergence of (1.7) depends on that of

$$\sum_{k=0}^{\infty} \varepsilon^k / \{ \varepsilon - c_m - \mathcal{R}(\alpha_m z) \}^k,$$

where $\varepsilon > 0$ is arbitrarily small, and there is convergence if $\mathcal{R}(ze^{-i\phi_m}) > c_m$, i.e. inside P. In particular, we have the following theorems.

THEOREM 3 [5]. *If $A(t)$ is an entire function of exponential type and if $G(z)$ is regular in a sufficiently large neighborhood of* 0, *then equation* (1.1) *has a solution regular in some neighborhood of* 0.

THEOREM 4. *If $A(t)$ is an entire function of exponential type zero and if $G(z)$ is regular at $z = 0$, then equation* (1.1) *has a solution regular in any bounded convex region containing* 0 *in which $g(z)$ is regular.*

With "the largest circle with centre at 0" instead of "any bounded convex region containing 0" this last theorem was given by Sheffer [5] and Muffgli [3]

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THE MAXIMUM TERM OF AN ENTIRE SEIRES, VI.

By S. M. SHAH

1. Introduction. Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of order ρ $(0 \leqslant \rho \leqslant \infty)$ and lower order λ. Let $\mu(r)$ denote the maximum term of the series for $|z| = r$ and $\nu(r)$ the rank of this term. Write

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(r)}{\nu(r)} = \begin{matrix} d \\ c \end{matrix}.$$

It is known that [1, p. 3]

$$c \leqslant 1/\rho \leqslant 1/\lambda \leqslant d. \tag{1.1}$$

(If we write $d^* = \limsup_{r\to\infty} \log \mu(r)/\nu(r)$, $c^* = \liminf_{r\to\infty} \log \mu(r)/\nu(r)$, then we have $c^* = c$ and $d^* \leqslant d$ with equality if $f(z)$ be of finite order. I have proved (1.1) with c and d replaced by c^* and d^* in [1, p. 80.])

If $f(z)$ be a function of finite order, then

$$\limsup_{r\to\infty} \log M(r)/\nu(r) \log r \leqslant 1 \tag{1.2}$$

the equality sign holding if [3]

$$\liminf_{r\to\infty} \log \log M(r)/\log \log r = 1. \tag{1.3}$$

The equality sign in (1.2) may hold even if (1.3) is not true. See Example 3.

When $0 < \rho < \infty$, write

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(r)}{r^\rho} = \begin{matrix} T \\ t \end{matrix}; \quad \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\nu(r)}{r^\rho} = \begin{matrix} \gamma \\ \delta \end{matrix}.$$

If $f(z)$ be of very regular growth order ρ, that is $0 < t < T < \infty$, then [2, p. 220] $0 < \delta < \gamma < \infty$ and it follows by an easy argument that

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$$\delta/\gamma\rho \leqslant c \leqslant d \leqslant \gamma/\delta\rho. \tag{1.4}$$

Hence for functions of perfectly regular growth order ρ, that is, $0 < t = T < \infty$, we have [2, p. 220] $0 < \delta = \gamma < \infty$ and $c = d = 1/\rho$.

If $\log M(r)/r^{\rho}$ tends to zero or infinity as r tends to infinity, then $\log M(r)/\nu(r)$ need not tend to a limit as r tends to infinity (see Example 1).

If $f(z)$ be of infinite order, the relation $\log M(r) \sim \log \mu(r)$ as $r \to \infty$ is not necessarily true and hence the inequality (1.2) may not hold. It is known that [3] $c = 0$ but d may have any value in the range $(1/\lambda, \infty)$. In fact given any number x, $0 \leqslant x \leqslant \infty$, we can construct a function of infinite order for which $d = x$. (See Example 2 and Theorem 3). We can also construct functions of infinite order for which $d^* = \limsup_{r\to\infty} \log \mu(r)/\nu(r) < d$.

2. Statement of results.

THEOREM 1. *If $f(z)$ be an entire function and if*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\log M(r)}{\log\log r} = \frac{L}{l}, \tag{2.1}$$

then we have

$$\limsup_{r\to\infty} \frac{\log M(r)}{\nu(r)} \frac{\log\log M(r)}{\log r} \geqslant \frac{1}{l}. \tag{2.2}$$

Further if $f(z)$ satisfies the relation $\log M(r) \sim \log \mu(r)$ *as* $r \to \infty$, *then*

$$\liminf_{r\to\infty} \frac{\log M(r)}{\nu(r)} \frac{\log\log M(r)}{\log r} \leqslant \frac{1}{L}. \tag{2.3}$$

THEOREM 2. *Given ρ $(0 \leqslant \rho < \infty)$ and any increasing function $\psi(x)$ such that*

$$\limsup_{x\to\infty} \log \psi(x)/\log x \leqslant \rho \tag{2.4}$$

there exists an entire function $f(z)$ of order ρ such that

$$\liminf_{r\to\infty} \psi(r) \log M(r)/\nu(r) = 0 \tag{2.5}$$

$$\limsup_{r\to\infty} \log M(r)/\nu(r) \log r = 1, \tag{2.6}$$

THEOREM 3. *Given any increasing function $\phi(x)$ tending to infinity (however rapidly) with x, there exist entire functions $f(z)$ and $F(z)$, both of infinite order, such that*

$$\liminf_{r\to\infty} \phi(r) \log M(r,f)/\nu(r,f) = 0 \tag{2.7}$$

$$\limsup_{r\to\infty} \log M(r,f)/\nu(r,f) \log r = 1 \tag{2.8}$$

and

$$\limsup_{r\to\infty} \log M(r,F)/\phi\{r\nu(r,F)\} = \infty. \tag{2.9}$$

3. **Proof of Theorem 1.** We first state a lemma.

LEMMA. *Let*

$$G(r) = \text{Constant} + \int_a^r \frac{g(x)}{x}\,dx, \tag{3.1}$$

where $g(x)$ is any function nondecreasing and positive for $x \geqslant a$, and let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log G(r)}{\log\log r} = \frac{L_1}{l_1},$$

then

$$\liminf_{r\to\infty} \frac{G(r)\log G(r)}{g(r)\log r} \leqslant \frac{1}{L_1} \leqslant \frac{1}{l_1} \leqslant \limsup_{r\to\infty} \frac{G(r)\log G(r)}{g(r)\log r}.$$

The proof is similar to that of Lemma of [4] and is omitted.

To prove the theorem let $g(x) = \nu(x)$ and write

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{lll\,\mu(r)}{ll\,r} = \frac{L_1}{l_1},$$

where $\log r = l_1 r = lr$, $\log\log r = l_2 r = ll\,r, \ldots,$ then we have from the lemma

$$\liminf_{r\to\infty} \frac{l\,\mu(r)\,ll\,\mu(r)}{\nu(r)\,lr} \leqslant \frac{1}{L_1} \leqslant \frac{1}{l_1} \leqslant \limsup_{r\to\infty} \frac{l\,\mu(r)\,ll\,\mu(r)}{\nu(r)\,lr}.$$

Since $\mu(r) \leqslant M(r)$, (2.2) follows, and since by hypothesis $\log M(r) \sim \log \mu(r)$, (2.3) follows.

REMARK (i) If we take $g(r) = n(r, a)$, where $n(r,a)$ denotes the number of zeros of $f(z)-a$ in $|z| \leqslant r$, we have from the lemma

$$\liminf_{r\to\infty} \frac{n(r,a)\log r}{lM(r)\,llM(r)} \leqslant l. \tag{3.2}$$

If $f(z)$ is of finite order then $l \leqslant 1$ and so the relation (3.2) includes as a special case the following result due to Valiron [5].

For every entire function of finite order we have

$$\liminf_{r\to\infty} \frac{n(r, a)}{lM(r)} \frac{\log r}{llM(r)} \leqslant 1.$$

(ii) We can similarly prove the following relations.

Let k ($\geqslant 2$) be an integer and a be any complex number. For every entire function we have

$$\liminf_{r\to\infty} \frac{l_1 \mu(r)\, l_2 \mu(r) \ldots l_k\, \mu(r)}{\nu(r)\, l_1 r \ldots \ldots l_{k-1} r} \leqslant 1 / \limsup_{r\to\infty} \frac{l_{k+1}\mu(r)}{l_k r} \quad (3.3).$$

$$\limsup_{r\to\infty} \frac{l_1 \mu(r) \ldots\ldots l_k\, \mu(r)}{\nu(r)\, l_1 r \ldots l_{k-1} r} \geqslant 1 / \liminf_{r\to\infty} \frac{l_{k+1}\, \mu(r)}{l_k\, r} \quad (3.4)$$

$$\liminf_{r\to\infty} \frac{n(r, a)\; l_1 r \ldots l_{k-1} r}{l_1 M(r) \ldots l_k M(r)} \sim \liminf_{r\to\infty} \sim \frac{l_{k+1}\, M(r)}{l_k\, r} \quad (3.5)$$

$$\limsup_{r\to\infty} \frac{n(r, a)\; l_1 r \ldots l_{k-1}\, r}{N(r, a)\; l_1\, N(r, a) \ldots l_{k-1} N(r, a)} \geqslant \limsup_{r\to\infty} \frac{l_k\, N(r, a)}{l_k\, r}$$

$$= \limsup_{r\to\infty} \frac{l_{k+1} M(r)}{l_k\, r} \quad (3.6)$$

except possibly for one value of a.

4. **Proof of Theorem 2.** We may suppose without loss of generality that $\log \psi(x) / \log x \geqslant \rho$ $(x \geqslant 2)$, since if (2.5) holds for a certain $\psi(r)$, it will hold for any $\psi_1(r)$ such that $\limsup_{r\to\infty} \psi_1(r)/\psi(r) < \infty$.

Hence $\log \psi(x) / \log x$ and $\log \psi(2x) / \log x$ tend to ρ as $x \to \infty$. Let

$$\lambda_1 = 1 + e^e,\ \lambda_{n+1} = \exp\ (\exp \lambda_n) \quad (4.1)$$
$$a_n = [\psi(2\lambda_n)\ (\log \lambda_n)^2] \quad (n = 1, 2, \ldots)$$

and
$$f(z) = \sum_{1}^{\infty} (z/\lambda_n)^{a_n}.$$

We can show that $f(z)$ is an entire function of order ρ. Further for all $n > n_o$, $a_n < a_{n+1}$.

For

$$r = r_n = \lambda_n \left(1 + \frac{1}{\psi(2\lambda_n)\, ll\lambda_n} \right) (n > n_1), \nu(r) = a_n$$

and

$$\frac{\psi(r) \log M(r)}{\nu(r)} \sim \frac{\psi(r_n)}{\psi(2\lambda_n)\, ll\lambda_n} < \frac{2}{ll\lambda_n} \to 0$$

as $n \to \infty$. Further for $r = \lambda_{n+1}$

$$\frac{\log M(r)}{\nu(r) \log r} \sim \frac{\log (\lambda_{n+1}/\lambda_n)}{\log \lambda_{n+1}} \to 1 \text{ as } n \to \infty.$$

5. **Proof of Theorem 3.** We may suppose that (i) $\phi(x) \geqslant 2$ for $x \geqslant 1$, (ii) $\log \phi(x)/\log x$ tends to infinity with x. Let λ_n be defined as in (4.1) and let β_n be a rapidly increasing sequence of positive integers such that $\beta_0 = 1$

$$\beta_n > \frac{2 \log \phi(2\lambda_n)}{\log \lambda_n} + \beta_{n-1}, \; c_n = [\exp(\beta_n \log \lambda_n)] \; (n = 1, 2, \ldots)$$

and

$$f(z) = \sum_{1}^{\infty} (z/\lambda_n)^{c_n}.$$

It is easy to see that $f(z)$ is an entire function of infinite order. Let

$$r_n = \lambda_n \left(1 + \frac{1}{\phi(2\lambda_n)\, ll\lambda_n} \right) = \lambda_n (1 + b_n),$$

say.

Then for $n > n_0$, $\nu(r_n, f) = c_n$ and

$$(1 + b_n)^{c_n} < M(r_n, f) < n[\lambda_n(1 + b_n)]^{c_n - 2} + [\lambda_n(1 + b_n)/\lambda_{n-1}]^{c_{n-1}} + (1 + b_n)^{c_n} + O(1). \quad (5.1)$$

Hence $\log M(r_n, f) \sim b_n \exp(\beta_n \log \lambda_n)$,

$$\frac{\phi(r_n) \log M(r_n, f)}{\nu(r_n, f)} < \frac{\phi(2\lambda_n)}{\phi(2\lambda_n)\, ll\lambda_n} \to 0 \text{ as } n \to \infty.$$

Further we can show as in (5.1) that $\log M(r, f) \sim \log \mu(r, f)$ and for $r = \lambda_{n+1}$

$$\frac{\log \mu(r, f)}{\nu(r, f) \log r} \sim \frac{c_n \log (\lambda_{n+1}/\lambda_n)}{c_n \log \lambda_{n+1}} \to 1 \text{ as } n \to \infty.$$

To prove (2.9) let

$$\xi_n = n^n, \ \mu_1 = 2, \ k_n = \left[\exp\left\{n\phi\left(\frac{\mu_n \xi_{n+1}}{\sqrt{n}}\right)\right\}\right],$$

$$\mu_{n+1} = [\exp (k_n + \mu_n + 1)] \quad n = 1, 2, 3, \ldots$$

$$\alpha_{n, m} = 1/(m+n)\ (m+\mu_n)$$

$$a_{n, m} = \frac{\xi_{n+1}}{\sqrt{n}} \quad \frac{1}{(1-\alpha_{n, m})}; \ \begin{pmatrix} m = 1, 2, \ldots k_n; \\ n = 1, 2, \ldots\ldots \end{pmatrix}$$

and

$$F(z) = \sum_1^\infty \left\{ \left(\frac{z}{\xi_n}\right)^{\mu_n} + \left(\frac{z}{a_{n,1}}\right)^{1+\mu_n} + \ldots\ldots + \left(\frac{z}{a_{n, k_n}}\right)^{k_n+\mu_n} \right\}.$$

The function $F(z)$ is an entire function of infinite order. Further for $r = \xi_{n+1}/\sqrt{n}$ we have $\nu(r, F) = \mu_n$ and

$$M(r, F) \geqslant \left\{1 - \frac{1}{(1+n)(1+\mu_n)}\right\}^{1+\mu_n} + \ldots$$
$$+ \left\{1 - \frac{1}{(k_n+n)(k_n+\mu_n)}\right\}^{k_n+\mu_n}$$
$$\geqslant e^{-1/n} + \ldots\ldots + e^{-1/(n+k_n-1)} > k_n\, e^{-1/n}.$$

Hence

$$\frac{\log M(r, F)}{\phi[r\, \nu(r, F)]} > \frac{\log k_n + o(1)}{\phi\left(\frac{\mu_n \xi_{n+1}}{\sqrt{n}}\right)} \to \infty$$

as $n \to \infty$, and the theorem is proved.

6. **Examples.** EXAMPLE 1. Let

$$n_1 = 10, \ n_{s+1} = n_s^4 \quad (s = 1, 2, \ldots\ldots),$$

$$R_1 = R_2 = \ldots = R_9 = 1,$$

$$R_m = m \log \log m \text{ for } n_s \leqslant m < n_s^4/\log n_s$$

$$= n_s^4 \log\log (n_s^4) - (n_{s+1} - m)/(n_s!)!,$$

for $$n_s^4/\log n_s \leqslant m < n_{s+1}\,;\; s = 1, 2, 3, \ldots$$

$$f(z) = 1 + \sum_1^\infty \frac{z^n}{R_1 R_2 \ldots\ldots R_n}.$$

Here $f(z)$ is an entire function of regular growth order 1 for which

$$\lim_{r\to\infty} \frac{\nu(r)}{r} = \lim_{r\to\infty} \frac{\log M(r)}{r} = 0,$$

$$\liminf_{r\to\infty} \frac{\log M(r)}{\nu(r)} = 0 \text{ and } \limsup_{r\to\infty} \frac{\log M(r)}{\nu(r)} = \infty.$$

REMARK. In order that $f(z)$ be of regular growth order ρ $(0 < \rho < \infty)$ it is sufficient but not necessary that $\log M(r)/\nu(r) \to 1/\rho$ as $r \to \infty$. (See Corollary [1, p. 80] where a correction is necessary in the statement.)

EXAMPLE 2. Let $0 \leqslant x < \infty$, $\lambda_{n+1} = (n+1)\exp(n\,x)$,

$$(n \geqslant 0),\; \mu_1 = 2,\; \mu_n = \Big[\exp\{\exp(\lambda_n + \mu_{n-1})\}\Big] (n \geqslant 2),$$

$$f(z) = \sum_1^\infty \left(\frac{z}{\lambda_n}\right)^{\mu_n}.$$

Here $f(z)$ is an entire function of infinite order such that

$$\lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \infty, \limsup_{r\to\infty} \frac{\log M(r)}{\nu(r)} = x.$$

EXAMPLE 3. Let $\lambda_1 = \lambda_2 = 2$, $\lambda_{n+1} = \lambda_n^{\log\log n}$ $(n \geqslant 3)$,

$$\epsilon_n = 1/\sqrt{\log \lambda_n},\; a_n = [\lambda_n^{\rho+\epsilon_n}]\; (n \geqslant 1,\; 0 \leqslant \rho < \infty)$$

and $$f(z) = \sum_1^\infty (z/\lambda_n)^{a_n}.$$

Then $f(z)$ is an entire function of order ρ for which

$$\limsup_{r\to\infty} \frac{\log M(r)}{\nu(r)\log r} = 1,\; \lim_{r\to\infty} \frac{ll\,M(r)}{ll\,r} = \infty.$$

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CERTAIN PROPERTIES OF THE GENERALIZED LAPLACE TRANSFORM

By S. K. Bose

1. The integral

$$\phi(p)=p\int_0^\infty (2\,x\,p)^{-\frac{1}{4}} W_{k,m}(2\,p\,x)\,f(x)dx \qquad (1)$$

is given by Dr. R. S. Varma [5]. $\phi(p)$ is called the *image* of $f(x)$ and $f(x)$ the *original* of $\phi(p)$, and the transform is known as the Whittaker transform and is symbolically denoted by [1]

$$\phi(p)\ \frac{k}{\overline{m}}\ f(x).$$

This reduces to the Laplace transform when $k=\frac{1}{4}$, $m=\pm\frac{1}{4}$, due to

$$(2\,x\,p)^{-\frac{1}{4}} W_{\frac{1}{4},\pm\frac{1}{4}}(2\,x\,p)\equiv e^{-xp}.$$

Other particular cases of this transform are as follows :

(i) When $k=\frac{1}{2}+\frac{1}{2}\,n+l$, $m=\pm\frac{1}{2}\,n$, the general transform is known as L_l^n –transform because

$$W_{\frac{1}{2}+\frac{1}{2}n+l,\,\pm\frac{1}{2}n}(x)=(-)^l e^{-\frac{1}{2}x} x^{\frac{1}{2}n+\frac{1}{2}} L_l^n(x),$$

l being a positive integer.

(ii) When $k=0$, the general transform is known as k-transform because

$$W_{0,m}(x)=\sqrt{(x/\pi)}\,K_m(x/2).$$

(iii) When $k=\frac{1}{2}n+\frac{1}{4}$, $m=\pm\frac{1}{4}$, the general transform is known as D_n– transform because

$$D_n(2\sqrt{x})=2^{n/2}(2x)^{-\frac{1}{4}} W_{\frac{1}{2}n+\frac{1}{4},\pm\frac{1}{4}}(2x).$$

In this paper the author obtains some new properties of this transform by using the properties of the Whittaker and Parabolic cylinder functions. The analogues of these properties do not exist in the case of the Laplace transform.

2. Theorem I*. *If*

$$\phi_{s+m+1/2}(p)\ \frac{m+s+\frac{1}{2}}{\overline{m}}\ f(x),$$

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* I am thankful to the referee for his suggestions.

then

$$\sum_{s=0}^{\infty}\frac{(-)^s\phi_{s+m+\frac{1}{2}}(p)}{s!\Gamma(2m+s+1)}=e(2p)^{m+\frac{1}{4}}\sum_{r=0}^{\infty}\frac{(-)^r(2p)^r}{r!\,\Gamma(2m+r+1)}\psi_{r+m+\frac{1}{4}}(p),$$

where

$$\psi_{r+m+\frac{1}{4}}(p)\doteqdot x^{m+r+\frac{1}{4}}f(x),$$

provided $R(p)>0$, $R(\mu\pm m+\frac{5}{4})>0$, *where* $f(x)=O(x^{\mu})$ *for small* x, $e^{-px}x^{\frac{1}{4}}J_{2m}(2\sqrt{px})f(x)$ *tends to zero as* $x\to\infty$ *and* $f(x)$ *is continuous for* $x>0$.

PROOF. We have

$$\phi_{s+m+\frac{1}{2}}(p)=p\int_0^{\infty}(2xp)^{-\frac{1}{4}}W_{s+m+\frac{1}{2},\,m}(2px)f(x)dx. \quad (1)$$

On multiplying by $\dfrac{(-)^s}{s!\,\Gamma(2m+s+1)}$, taking the sum from zero to infinity, and using [3]

$$et^{\frac{1}{2}}e^{-\frac{1}{2}t}J_{2m}(2t^{\frac{1}{2}})=\sum_{s=0}^{\infty}\frac{(-)^sW_{s+m+\frac{1}{2},\,m}(t)}{s!\,\Gamma(2m+s+1)},$$

we get

$$\sum_{s=0}^{\infty}\frac{(-)^s}{s!\,\Gamma(2m+s+1)}\phi_{s+m+\frac{1}{2}}(p)$$
$$=pe\int_0^{\infty}(2xp)^{-\frac{1}{4}}f(x)\,(2xp)^{\frac{1}{2}}e^{-px}J_{2m}\{2\sqrt{(px)}\}\,dx. \quad (2)$$

Further, on using

$$J_\nu(z)=\frac{1}{\Gamma(\nu+1)}(z/2)^{\nu}{}_0F_1[\nu+1,\ -\tfrac{1}{4}z^2],$$

we obtain

$$\sum_{s=0}^{\infty}\frac{(-)^s\phi_{s+m+\frac{1}{2}}(p)}{s!\,\Gamma(2m+s+1)}=e(2p)^{m+\frac{1}{4}}$$
$$\times\sum_{r=0}^{\infty}\frac{(-)^r(2p)^r}{r!\,\Gamma(2m+r+1)}\psi_{r+m+\frac{1}{4}}(p),$$

where

$$\psi_{r+m+\frac{1}{4}}(p) \doteqdot x^{r+m+\frac{1}{4}} f(x),$$

provided $R(p) > 0$ and $R(\mu \pm m + \frac{5}{4}) > 0$, where $f(x) = O(x^{\mu})$ for small x and $e^{-px} x^{\frac{1}{4}} J_{2m}\{2\sqrt{(2px)}\}$ tends to *zero* uniformly as $x \to \infty$.

Now, to justify the change of order in (2), we see [2, p. 500] that

$$\text{(i)} \quad \sum_{s=0}^{\infty} \frac{(-)^s}{s!\,\Gamma(2m+s+1)} W_{s+m+\frac{1}{2}, m}(2px)$$
$$= e(2px)^{\frac{1}{2}} e^{-px} J_{2m}\{2\sqrt{(2px)}\}$$

is uniformly convergent in any fixed interval $0 \leqslant x \leqslant b$, b being arbitrary;

(ii) $f(x)$ is continuous for all values of $x > 0$; and

$$\text{(iii)} \quad \int_0^{\infty} e^{-px} |(2xp)^{\frac{1}{4}} J_{2m}\{2\sqrt{(2px)}\} f(x)|\, dx$$

is convergent, if $R(\mu \pm m + \frac{5}{4}) > 0$, where $f(x) = O(x^{\mu})$ for small x, and $e^{-px} x^{\frac{1}{4}} J_{2m}\{2\sqrt{(px)}\} f(x)$ tends to zero as $x \to \infty$.

The second change of order of integration and summation can be justified in the same way as above.

EXAMPLES.

(i) Let $\qquad f(x) = x^n$

then

$$\psi_{r+m+\frac{1}{4}}(p) = \frac{\Gamma(m+n+r+\frac{5}{4})}{p^{m+n+r+\frac{1}{4}}}$$

and [1, p. 12 (1)]

$$\phi_{s+m+\frac{1}{2}}(p) = \frac{\Gamma(n+m+\frac{5}{4})\,\Gamma(n-m+\frac{5}{4})}{p^n 2^{n+1} \Gamma(n-m-s+\frac{5}{4})}\, {}_2F_1\left\{\begin{matrix} n+m+\frac{5}{4}, n-m+\frac{5}{4} \\ n-s-m+\frac{5}{4} \end{matrix}, \frac{1}{2}\right\}.$$

Hence, by Theorem I, we obtain

$$\frac{\Gamma(2m+1)\,\Gamma(n-m+\frac{5}{4})}{e 2^{n+m+\frac{5}{4}}} \sum_{s=0}^{\infty} \frac{(-)^s}{s!\,\Gamma(2m+s+1)\,\Gamma(n-m-s+\frac{5}{4})}$$
$$\times {}_2F_1\left\{\begin{matrix} n+m+\frac{5}{4}, n-m+\frac{5}{4} \\ n-s-m+\frac{5}{4} \end{matrix}; \frac{1}{2}\right\}$$

$= {}_1F_1[m+n+\frac{5}{4};\ 2m+1;\ -2],\ R(p) > 0$ and $R(n \pm m+\frac{5}{4}) > 0.$

(ii) Let [1, p. 13 (3)]

$$f(x) = x^{\nu-1}e^{-ax} \overset{s+m+\frac{1}{2}}{\underset{m}{=}} \frac{\Gamma(\nu+m+\frac{1}{4})\,\Gamma(\nu-m+\frac{1}{4})}{2(2p)^{\nu-1}\,\Gamma(\nu-m-s+\frac{1}{4})}$$

$$\times {}_2F_1\left\{ \begin{matrix} \nu+m+\frac{1}{4},\ \nu-m+\frac{1}{4} \\ \nu-m-s+\frac{1}{4} \end{matrix},\ \frac{1}{2}-\frac{a}{2p} \right\}$$

$$= \phi_{s+m+\frac{1}{2}}(p),$$

$R(\nu \pm m+\frac{1}{4}) > 0,\ R(p) > 0$ and $|p| > |a|$, and [5, p. 16]

$$\psi_{r+m+\frac{1}{4}}(p) = \frac{p\Gamma(\nu+r+m+\frac{1}{4})}{(p+a)^{\nu+m+r+\frac{1}{4}}}.$$

Hence, from Theorem I, we obtain

$$e\left\{2p(p+a)^{-1}\right\}^{\nu+m+\frac{1}{4}}{}_1F_1[\nu+m+\tfrac{1}{4};\ 2m+1,\ -2p(p+a)^{-1}]$$

$$= \Gamma(2m+1)\,\Gamma(\nu-m+\tfrac{1}{4}) \sum_{s=0}^{\infty} \frac{(-)^s}{s!\,\Gamma(2m+s+1)\,\Gamma(\nu-m-s+\frac{1}{4})}$$

$$\times {}_2F_1\left\{ \begin{matrix} \nu+m+\frac{1}{4},\ \nu-m+\frac{1}{4} \\ \nu-m-s+\frac{1}{4} \end{matrix},\ \frac{1}{2}-\frac{a}{2p} \right\},$$

$$R(\nu \pm m+\tfrac{1}{4}) > 0,\ R(p) > 0 \text{ and } |p| > |a|. \tag{4}$$

3. Theorem II. *If* $\phi_n(p) \overset{\frac{1}{2}n+\frac{1}{4}}{\underset{\pm\frac{1}{4}}{=}} f(x)$ *and* $\psi(p) \doteqdot x^{-\frac{1}{2}}f(x)$ *then*

$$2\sqrt{2p} \sum_{n=0}^{\infty} \frac{(-)^n 2^{2n} n!}{(2n+1)!} \varphi_{2n+1}(p) = \psi(p),$$

provided $R(p) > 0,\ R(\mu) > -\frac{1}{2}$, *where* $f(x) = O(x^{\mu})$ *for small* x, $x^{\frac{1}{2}}e^{-px}f(x)$ *tends to zero as* $x \to \infty$ *and* $f(x)$ *is continuous for* $x > 0$.

Proof. We have

$$\phi_{2n+1}(p) = 2^{-n-\frac{1}{2}}p \int_0^{\infty} D_{2n+1}\left\{2\sqrt{(px)}\right\} f(x)dx. \tag{1}$$

On multiplying by $\dfrac{(-)^n 2^n n!}{(2n+1)!}$ and taking the sum from zero to infinity, we get

$$\sqrt{2}\sum_{n=0}^{\infty}\frac{(-)^n 2^{2n} n!}{(2n+1)!}\phi_{2n+1}(p)$$

$$=p\int_0^{\infty} f(x)\sum_{n=0}^{\infty}\frac{(-)^n 2^n n!}{(2n+1)!}D_{2n+1}\{2\sqrt{(px)}\}dx \qquad (2)$$

and using [6]

$$\frac{e^{-\frac{1}{2}x}}{\sqrt{(2x)}}=\sum_{n=0}^{\infty}\frac{(-)^n 2^n n!}{(2n+1)!}D_{2n+1}\{\sqrt{(2x)}\},$$

we obtain

$$\sqrt{2}\sum_{n=0}^{\infty}\frac{(-)^n 2^{2^n} n!}{(2n+1)!}\varphi_{2n+1}(p)=\frac{1}{2\sqrt{p}}\psi(p),$$

where $\qquad \psi(p) \doteqdot x^{-\frac{1}{2}}f(x).$

Now, in order to justify the change of order [2, p. 500] in (2), we see that

$$\text{(i)} \qquad e^{-\frac{1}{2}x}=\sqrt{(2x)}\sum_{n=0}^{\infty}\frac{(-)^n 2^n n!}{(2n+1)!}D_{2n+1}\{2\sqrt{(px)}\}$$

is uniformly convergent in any fixed interval $0 \leqslant x \leqslant b$, b arbitrary,

(ii) $f(x)$ is continuous for all values of $x > 0$; and

(iii) $\int_0^{\infty} e^{-px}\,|x^{-\frac{1}{2}}f(x)|\,dx$ is convergent, if $R(\mu) > -\frac{1}{2}$, where $f(x)=O(x^{\mu})$ and $x^{\frac{1}{2}}e^{-px}f(x)$ tends to zero as $x \to \infty$.

Examples. (i) Let [1, p. 14 (7)]

$$f(x)=x^{\nu/2}J_\nu\{2\sqrt{(x)}\} \doteqdot \frac{\frac{1}{2}^{n+\frac{1}{4}}}{\pm\frac{1}{4}}\tfrac{1}{2}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma(\nu+r+\frac{3}{2})}{r!\,\Gamma(\nu-\frac{1}{2}n+r+\frac{3}{2})\,(2p)^{\nu+r}}$$

$$\times{}_2F_1\left\{\begin{matrix}\nu+r+\frac{3}{2},\ \nu+r+1\\ \nu-\frac{1}{2}n+r+\frac{3}{2}\end{matrix},\ \frac{1}{2}\right\}$$

$$=\phi_n(p), \qquad R(\nu) > -1 \text{ and } R(p) > 0,$$

and [4, p. 29]

$$\psi(p)=\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)\,p^{\nu-\frac{1}{2}}}\,{}_1F_1[\nu+\tfrac{1}{2};\ \nu+1;\ -1/p],\ R(\nu) > -\tfrac{1}{2}.$$

Hence, from Theorem II, we obtain

$$\frac{2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\,{}_1F_1[\nu+\tfrac{1}{2};\ \nu+1;\ -1/p]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-)^{n+r}2^{2n}\Gamma(n+1)\Gamma(\nu+r+\frac{3}{2})}{(2n+1)!\,r!\,\Gamma(\nu-n+r+1)(2p)^r}$$

$$\times{}_2F_1\left\{\begin{matrix}\nu+r+\frac{3}{2},\ \nu+r+1\\ \nu-n+r+1\end{matrix};\ \frac{1}{2}\right\}$$

$$R(\nu)>-\tfrac{1}{2}\text{ and }R(p)>0. \qquad (3)$$

The convergence of the right side follows from the absolute convergence of the n-series and the r-series.

(ii) Let [1, p. 13 (3)] $f(x)=x^{\lambda}e^{-qx}$

$$\frac{\frac{1}{2}n+\frac{1}{4}}{\pm\frac{1}{4}}\ \frac{\Gamma(\lambda+\frac{3}{2})\Gamma(\lambda+1)}{2(2p)^{\lambda}\Gamma(\lambda-\frac{1}{2}n+\frac{3}{2})}\,{}_2F_1\left\{\begin{matrix}\lambda+\frac{3}{2},\ \lambda+1\\ \lambda-\frac{1}{2}n+\frac{3}{2}\end{matrix};\ \frac{1}{2}-\frac{q}{2p}\right\}$$

$$=\phi_n(p),\ R(\lambda)>-1,\ R(p)>0,\ |p|>|q|,$$

and [4, p. 16] $\psi(p)=\dfrac{\Gamma(\lambda+\frac{1}{2})p}{(p+q)^{\lambda+\frac{1}{2}}},\ R(\lambda)>-\frac{1}{2}$.

Hence, from Theorem II, we obtain

$$\{2p(p+q)^{-1}\}^{\lambda+\frac{1}{2}}=(2\lambda+1)\Gamma(\lambda+1)\sum_{n=0}^{\infty}\frac{(-)^{n}2^{2n}n!}{(2n+1)!\,\Gamma(\lambda-n+1)}$$

$$\times{}_2F_1\left\{\begin{matrix}\lambda+\frac{3}{2},\ \lambda+1\\ \lambda-n+1\end{matrix};\ \frac{1}{2}-\frac{q}{2p}\right\}$$

$$R(\lambda)>-\tfrac{1}{2},\ R(p)>0\text{ and }|p|>|q|. \qquad (4)$$

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ON TSCHEBYSCHEFF'S FUNCTION $T_n(z)$ AND ITS ASSOCIATED EQUATIONS

By HARIDAS BAGCHI AND NALINI KANTA CHAKRAVARTY

1. The function $T_n(z)$ introduced by Tschebyscheff in 1850 in *Mem. Acad. Petersburg* is one solution of the differential equation

$$(1-z^2)\frac{d^2w}{dz^2}-z\frac{dw}{dz}+n^2w=0 \tag{I}$$

and of the functional equation

$$f_{n+1}(z)-zf_n(z)+\tfrac{1}{4}f_{n-1}(z)=0. \tag{II}$$

Assuming the general solution of (I) in the form

$$w=\sum_{r=0}^{\infty}a_r z^r. \tag{1}$$

we readily obtain

$$a_{r+2}=\frac{r^2-n^2}{(r+2)(r+1)}a_r, \tag{2}$$

so that

$$a_{n+2}=a_{n+4}=\ldots=a_{n+2k}=0. \tag{3}$$

Choosing $a_n=1$, $a_{n-1}=0$, we obtain the polynomial solution

$$w=T_n(z)=z^n-\frac{n(n-1)}{2(2n-2)}z^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.\,4.\,(2n-2)(2n-4)}z^{n-4}-\ldots, \tag{4}$$

$$=\frac{1}{2^{n-1}}\cos(n\cos^{-1}z). \tag{5}$$

Also,

$$T_n(z)=\frac{(-1)^{n/2}}{2^{n-1}}.\quad F\left(-\frac{n}{2},\frac{n}{2},\frac{1}{2};z^2\right), \tag{6}$$

or,

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$$\frac{(-1)^{(n-1)/2} n.z}{2^{n-1}} F\left(\frac{1-n}{2}, \frac{1+n}{2}, \frac{3}{2}; z^2\right) \tag{7}$$

according as n is even or odd $T_0(z)$ is taken as unity.

We may also obtain a solution of (I) in negative powers of z. Taking

$$w = \sum_{r=0}^{\infty} \frac{b_r}{z^r}, \tag{8}$$

and imposing the conditions: $b_n = \frac{1}{2^{2n}}, b_{n-1} = 0$, we obtain the solution

$$w = \frac{1}{2^{2n}} \left[\frac{1}{z^n} + \frac{n(n+1)}{4.1!(1+n)} \cdot \frac{1}{z^{n+2}} + \frac{n(n+1)(n+2)(n+3)}{4^2.2!(1+n)(2+n)} \cdot \frac{1}{z^{n+4}} + \cdots, \right] \tag{9}$$

$$= \frac{1}{2^{2n} z^n} F\left(\frac{n}{2}, \frac{n+1}{2}, 1+n; \frac{1}{z^2}\right). \tag{10}$$

We shall denote this function by $V_n(z)$ and call it *Tschebyscheff's function of the second kind.* The series is absolutely and uniformly convergent, when $|z| > 1$. The general solution of (I), valid outside the unit circle $|z| = 1$, is given by

$$w = A\,T_n(z) + B\,V_n(z). \tag{11}$$

2. It is well known that $T_n(z)$ satisfies the recurrence relation (II), for $n \geqslant 2$.* In order to verify that $V_n(z)$ is also a solution of (II), we note that when $f_n(z)$ is replaced by $V_n(z)$, the coefficient of $z^{-(n+2r-1)}$ $(r \geqslant 1)$ in the left side of (*II*) is

$$\lambda \left[\frac{1}{n+r} - \frac{n(n+2r-1)}{r(1+n)(r+n)} + \frac{n(n-1)}{rn(n+1)} \right] = 0,$$

where

$$\lambda = \frac{(n+1)(n+2)\ldots(n+2r-2)}{4^{r+n}(r-1)!\,(n+2)(n+3)\ldots(n+r-1)}.$$

* For $n=1$, we have, however, $T_2(z) - z\,T_1(z) + \frac{1}{4}\,T_0(z) = -\frac{1}{4}$.

The coefficient of $z^{-(n-1)}$ (corresponding to $r=0$) is also zero. Hence $V_n(z)$ satisfies (II). The general solution of (II) is therefore

$$f_n(z) = G(z)\, T_n(z) + H(z)\, V_n(z), \tag{12}$$

where $G(z)$ and $H(z)$ are arbitrary functions.

3. Writing (II) in the form

$$\frac{f_{n+1}(z)}{f_n(z)} = z - \frac{\frac{1}{4}}{\dfrac{f_n(z)}{f_{n-1}(z)}}, \tag{13}$$

we obtain the finite continued fraction

$$\frac{T_{n+1}(z)}{T_n(z)} = z - \frac{\frac{1}{4}}{z-}\,\frac{\frac{1}{4}}{z-}\cdots\frac{\frac{1}{4}}{z-}\,\frac{1}{2z}. \tag{14}$$

4. If $\alpha_n(z)$ and $\beta_n(z)$ be any two particular solutions of (II), we have

$$\alpha_{n+1}(z) - z\,\alpha_n(z) + \tfrac{1}{4}\alpha_{n-1}(z) = 0 \tag{15}$$

and

$$\beta_{n+1}(z) - z\,\beta_n(z) + \tfrac{1}{4}\beta_{n-1}(z) = 0. \tag{16}$$

Multiplying (15) and (16) by $\beta_n(z)$ and $\alpha_n(z)$ respectively and subtracting, we obtain

$$\gamma_{n+1}(z) = \tfrac{1}{4}\gamma_n(z), \tag{17}$$

where

$$\gamma_r(z) = \alpha_r(z)\,\beta_{r-1}(z) - \alpha_{r-1}(z)\,\beta_r(z). \tag{18}$$

Hence,

$$\gamma_n(z) = \frac{1}{4^{n-1}}\,\gamma_1(z). \tag{19}$$

In particular, we obtain

$$T_n(z)\,V_{n-1}(z) - T_{n-1}(z)\,V_n(z) = \frac{1}{4^{n-2}}\left[T_2(z)\,V_1(z) - T_1(z)\,V_2(z)\right]$$

$$= \frac{1}{2^{2n-3}}\left[(2z^2-1)\,V_1(z) - 2z\,V_2(z)\right], \text{ for } n \geqslant 2. \tag{20}$$

5. Multiplying (II) by h^{n+1}, we obtain (postponing the question of convergence),

$$\sum_{n=2}^{\infty} h^{n+1} f_{n+1}(z) - zh \sum_{n=2}^{\infty} h^n f_n(z) + \tfrac{1}{4} h^2 \sum_{n=2}^{\infty} h^{n-1} f_{n-1}(z) = 0.$$

Considering the two cases $n \geqslant 2$ and $n \geqslant 1$, and writing $h = 2t$, we obtain

$$\sum_{n=0}^{\infty} 2^n t^n f_n(z) = \frac{(1-2zt+t^2) f_0(z) + 2t(1-2zt) f_1(z) + 4t^2 f_2(z)}{1-2zt+t^2}, \tag{21}$$

or,

$$\sum_{n=0}^{\infty} 2^n t^n f_n(z) = \frac{(1-2zt) f_0(z) + 2t f_1(z)}{1-2zt+t^2}, \tag{22}$$

according as (II) is true for $n \geqslant 2$ or for $n \geqslant 1$. In particular, we have [3]

$$\sum_{n=0}^{\infty} 2^n t^n T_n(z) = \frac{1-t^2}{1-2zt+t^2} \tag{23}$$

and

$$\sum_{n=0}^{\infty} 2^n t^n V_n(z) = \frac{1-2zt+2tV_1(z)}{1-2zt+t^2}. \tag{24}$$

Employing a mode of reasoning similar to that used in a previous paper [1], one can easily show that the formulae (21)-(22) require for their validity that $|t| < 1$, and that the point z should be inside the domain Ω of the ellipse having the points $+1$ and -1 for foci, and having $|t| + |t^{-1}|$ as the length of the major axis. If however $f_n(z)$ possesses singularities, the domain of validity will be that portion of Ω over which the set of functions $f_n(z)$ is analytic. Thus, for instance, the series $\sum_{n=0}^{\infty} 2^n t^n V_n(z)$ has as its domain of convergence, the portion of the Argand plane exterior to the unit circle $|z| = 1$, but interior to the ellipse $|z-1| + |z+1| = |t| + |t^{-1}|$.

6. Let $\psi_n(z)$ and $\phi_n(z)$ be two particular analytic solutions of (II), so that

$$\psi_{n+1}(x)-x\psi_n(x)+\tfrac{1}{4}\psi_{n-1}(x)=0 \qquad (25)$$

$$\phi_{n+1}(y)-y\phi_n(y)+\tfrac{1}{4}\phi_{n-1}(y)=0. \qquad (26)$$

Multiplying (25) and (26) by $\phi_n(y)$ and $\psi_n(x)$ respectively and subtracting, we easily obtain by summation

$$\sum_{r=1}^{n} 4^{r-1}\psi_r(x)\phi_r(y)=\frac{1}{4(y-x)}\left[\psi_1(x)\phi_0(y)-\psi_0(x)\phi_1(y)\right] -\bar{R}_n(x,y), \qquad (27)$$

where

$$\bar{R}_n(x,y)=4^{n-1}\left[\frac{\psi_{n+1}(x)\phi_n(y)-\psi_n(x)\phi_{n+1}(y)}{y-x}\right].$$

Taking $\psi=\phi$, we have

$$\sum_{r=1}^{n} 4^{r-1}\psi_r(x)\psi_r(y)=\frac{1}{4(y-x)}\left[\psi_1(x)\psi_0(y)-\psi_0(x)\psi_1(y)\right] -R_n(x,y),$$

where

$$R_n(x,y)=4^{n-1}\frac{\left[\psi_{n+1}(x)\psi_n(y)-\psi_n(x)\psi_{n+1}(y)\right]}{y-x}. \qquad (28)$$

Making $y\to x$, we obtain

$$\sum_{r=1}^{n} 4^{r-1}\{\psi_r(x)\}^2=\tfrac{1}{4}\left[\psi_1(x)\,\psi_0'(x)-\psi_0(x)\,\psi_1'(x)\right]-R_n(x,x),$$

where

$$R_n(x,x)=4^{n-1}\left[\psi_{n+1}(x)\psi_n'(x)-\psi_n(x)\,\psi'_{n+1}(x)\right]. \qquad (29)$$

The reader will write down for himself the particular cases when ψ and ϕ are given the values T_r and V_r.

It may be interesting to discuss the circumstances, under which R_n and $\bar{R}_n$ tend to zero as n tends to infinity, i.e. the condition of convergence of the left side of (27)—(29) when $n\to\infty$.

7. From the formula (23) and the well-known relation

$$(1-2zt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(z), \tag{30}$$

we obtain

$$\sum_{n=0}^{\infty} 2^r t^r T_r(z) = (1-t^2)\left\{\sum_{n=0}^{\infty} t^n P_n(z)\right\}^2.$$

Equating the coefficients of t^r on both sides, we obtain the following relation between $T_r(z)$ and Legendre polynomials

$$2^r T_r(z) = \sum_{m+n=r}\sum P_m(z)\,P_n(z) - \sum_{m+n=r-2}\sum P_m(z)\,P_n(z). \tag{31}$$

The formula (10) associated with the integral formula $F(a,b,c\,;x)$

$$= \frac{\Pi(c-1)}{\Pi(b-1)\,\Pi(c-b-1)}\int_0^1 v^{b-1}(1-v)^{c-b-1}(1-vx)^{-a}dv$$

leads to the result

$$V_n(z) = \frac{1}{4^n z^n}\,\frac{\Pi(n)}{\{\Pi(n-1)/2\}^2}\int_0^1 v^{\frac{n-1}{2}}(1-v)^{\frac{n-1}{2}}(1-vz^{-2})^{-\frac{n}{2}}dv$$

$$= \frac{1}{2^{2n-1}}\,\frac{\Pi(n)}{\{\Pi(n-1)/2\}^2}\int_0^{\pi/2}\frac{\sin^n\theta\cos^n\theta}{(z^2-\sin^2\theta)^{n/2}}d\theta. \tag{32}$$

8. The functions $T_m(z)$ satisfy the quasi-orthogonal relation [3, 87 Ex. 1.]

$$\int_{-1}^{+1}\frac{T_m(z)\,T_n(z)}{(1-z^2)^{\frac{1}{2}}}\,dz = 0,\ m\neq n \tag{33}$$

$$\int_{-1}^{1}\frac{\{T_n(z)\}^2}{(1-z^2)^{\frac{1}{2}}}dz = \frac{\pi}{2^{2n-1}}. \tag{34}$$

Let us now consider the most general polynomial $f_n(z)$ such that

$$\int_{-1}^{1}\frac{\phi_m(z)\,f_n(z)}{(1-z^2)^{\frac{1}{2}}}\,dz = 0 \tag{35}$$

for *all arbitrary polynomials* $\phi_m(z)$ of degree $m<n$.

Any polynomial of degree m can be expressed uniquely as a linear combination of $m+1$ algebraic polynomials of degrees $\leqslant m$, at least one of which is of degree m. We can therefore write

$$\phi_m(z) \equiv \sum_{r=0}^{m} a_r T_r(z)$$

$$f_n(z) \equiv \sum_{s=0}^{n} b_s T_s(z).$$

From (33), (34), (35) we readily obtain

$$\sum_{r=0}^{m} a_r b_r \lambda_r = 0, \text{ where none of the } \lambda\text{'s vanishes.}$$

Since the a_r's are arbitrary, it follows $b_r = 0$, $(r = 0, 1, \ldots, m)$.

Hence $f_n(z) = b_n T_n(z)$, since m can be taken as large as $n-1$. The condition

$$f_n(0) = \frac{1}{2^{n-1}} \cos\left(\frac{n\pi}{2}\right) \tag{36}$$

would identify $f_n(z)$ with $T_n(z)$.

A slightly different method is by putting $z = \cos\theta$ in (35). We get

$$\int_{-\pi}^{\pi} \phi_m(\cos\theta) f_n(\cos\theta) d\theta = 0.$$

Since $\cos^n\theta$ is expressible as a linear combination of $\cos n\theta$, $\cos(n-2)\theta$, $\cos(n-4)\theta, \ldots$, we can take
$f_n(\cos\theta) = a_n \cos n\theta + a_{n-1} \cos(n-1)\theta + a_{n-2} \cos(n-2)\theta + \ldots$
and similarly for $\phi_m(\cos\theta)$. Proceeding as before, we obtain $a_r = 0$ $(r = 0, 1, \ldots, n-1)$.

Hence

$$f_n(\cos\theta) = a_n \cos(n \cos^{-1} z),$$

which reduces to $T_n(z)$ if the condition (36) is imposed.

The Tschebyscheff polynomial $T_n(z)$ can therefore be defined as the function $f_n(z)$ which satisfies (35) and (36), for arbitrary $\phi(z)$, $(m < n)$. For a parallel definition of the Legendre function $P_n(z)$, see [2] and [4].

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SOME FURTHER RELATIONS CONNECTED WITH TSCHEBYSCHEFF'S FUNCTION $T_n(z)$

By HARIDAS BAGCHI AND NALINI KANTA CHAKRAVARTY

1. In this article we propose to work out four relations (A), (B), (C), (D) that connect Tschebyscheff's function $T_n(z)$ with the Legendre function $P_n(z)$.

Let us write

$$U \equiv (1-2hz+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(z) \tag{1}$$

and

$$S \equiv \frac{1-h^2}{1-2hz+h^2} = \sum_{n=0}^{\infty} 2^n h^n T_n(z), \tag{2}$$

the variables z, h (real or complex) conforming to the relation:

$$|h| < |z \pm \sqrt{z^2-1}|, \text{ i. e. } |z+1|+|z-1| < |h| + \frac{1}{|h|}. \tag{3}$$

Subject to the relation (3), the series on the right side of (1) and (2) are uniformly convergent [1] w.r.t. z and h so as to be differentiable termwise w.r.t. each of them.

Then using the relation

$$(1-h^2)\frac{\partial U}{\partial x} = hUS, \tag{4}$$

we ultimately arrive at the equality

$$(1-h^2) \sum_{n=0}^{\infty} h^n P_n'(z) = h. \sum_{n=0}^{\infty} h^n P_n(z) . \sum_{n=0}^{\infty} 2^n h_n T_n(z).$$

Equating coefficients of h^n, we readily obtain

$$P'_n(z) - P'_{n-2}(z) = \sum_{r=1}^{n} 2^{r-1}. \; T_{r-1}(z) P_{n-r}(z). \tag{A}$$

[The result (A) can be put in a neater form* by using the following recurrence relation for $P_n(z)$:

$$P'_{n+1}(z) - P'_{n-1}(z) = (2n+1) P_n(z). \tag{I}$$

Thus changing n into $n+1$ in (A) and using (I), we get

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* Suggested by the referee.

$$(2n+1)P_n(z) = \sum_{1}^{n+1} 2^{r-1} T_{r-1}(z) P_{n+1-r}(z)$$

$$= \sum_{0}^{n} 2^r T_r(z) P_{n-r}(z), \text{ changing } r-1 \text{ to } r$$

$$= \sum_{0}^{n} 2^{n-r} T_{n-r}(z) P_r(z), \text{ writing in the}$$

reverse order.

This simplifies to

$$nP_n(z) = \sum_{0}^{n-1} 2^{n-1-r} T_{n-r}(z) P_r(z)$$

$$= \sum_{0}^{n-1} 2^r T_{r+1}(z) P_{n-1-r}(z), \text{ reversing the order.}$$

Changing again n into $(n+1)$ we get finally

$$(n+1)\, P_{n+1}(z) = \sum_{0}^{n} 2^r T_{r+1}(z) P_{n-r}(z). \qquad (A')]$$

Again squaring (1) and making use of (2), we get

$$(1-h^2)U^2 = S.$$

Logarithmic partial differentiation of this relation w.r.t. z gives

$$2S\frac{\partial U}{\partial z} = U\frac{\partial S}{\partial z}, \qquad (5)$$

or,

$$2\sum_{n=0}^{\infty} 2^n h^n T_n(z) \sum_{n=0}^{\infty} h^n P_n'(z) = \sum_{n=0}^{\infty} h^n P_n(z) . \sum_{n=0}^{\infty} 2^n h^n T_n'(z),$$

whence equating co-efficients of h^{n-1}, we have

$$\sum_{r=1}^{n} 2^r T_{r-1}(z) P'_{n-r}(z) = \sum_{r=1}^{n} 2^{n-r} P_{r-1}(z)\, T'_{n-r}(z),$$

i.e.

$$\sum_{r=1}^{n-1} 2^r T_{r-1}(z) P'_{n-r}(z) = \sum_{r=1}^{n-1} 2^{n-r} P_{r-1}(z)\, T'_{n-r}(z), \qquad (B)$$

for $P_0(z)$ and $T_0(z)$ are mere constants.

Further, from (1) we easily derive

$$h\frac{\partial U}{\partial h} = (z-h)\frac{\partial U}{\partial z}. \qquad (6)$$

Eliminating $\partial U/\partial z$ from (4) and (6), we have

$$(1-h^2)\frac{\partial U}{\partial h} = (z-h)US, \tag{7}$$

or,

$$(1-h^2)\sum_{n=0}^{\infty} nh^{n-1}P_n(z) = (z-h)\sum_{n=0}^{\infty} h^nP_n(z)\sum_{n=0}^{\infty} 2^nh^nT_n(z),$$

whence equating co-efficients of h^{n-1},

$$nP_n(z)-(n-2)P_{n-2}(z) = z\sum_{r=1}^{n} 2^{r-1}P_{n-r}(z)T_{r-1}(z)$$
$$-\sum_{r=1}^{n-1} 2^{r-1}P_{n-r-1}(z)T_{r-1}(z). \tag{C}$$

Also eliminating $\partial U/\partial z$ from (5) and (6), we get

$$2hS\frac{\partial U}{\partial h} = (z-h)U\frac{\partial S}{\partial z} \tag{8}$$

or,

$$2h.\sum_{n=0}^{\infty} 2^nh^nT_n(z).\sum_{n=0}^{\infty} nh^{n-1}P_n(z)$$
$$= (z-h).\sum_{n=0}^{\infty} h^nP_n(z).\sum_{n=0}^{\infty} 2^nh^nT'_n(z).$$

Equating co-efficients of h^{n+1}, we derive

$$\sum_{r=1}^{n+1} 2^{n-r+2}rP_r(z)T_{n-r+1}(z) = z\sum_{r=1}^{n+1} 2^rT'_r(z)P_{n-r+1}(z)$$
$$-\sum_{r=1}^{n} 2^rT'_r(z)P_{n-r}(z). \tag{D}$$

2. If we now set

$$f(t) \equiv (1-t^2)^{(2n-1)/2}$$

and appeal to the formula for the nth derivative of an analytic function, viz.

$$f^{(n)}(z) = \frac{n!}{(2\pi i)}\int_c \frac{f(t)}{(t-z)^{n+1}}dt,$$

we are led to

$$\frac{d^n}{dz^n}\cdot\left[(1-z^2)^{(2n-1)/2}\right]=\frac{(-1)^{(2n-1)/2}.\ n!}{2\pi i}\int_C\frac{(t^2-1)^{(2n-1)/2}}{(t-z)^{n+1}}\,dt.$$

Consequently

$$\frac{T_n(z)}{\sqrt{z^2-1}}=\frac{(n!)^2}{(2n)!\,\pi i}\int_c\frac{(t^2-1)^{(2n-1)/2}}{(t-z)^{n+1}}\,dt, \tag{9}$$

where C is a simple closed contour encircling the point z.

Manifestly, the formula (4) corresponds to Schlafli's integral formula for $P_n(z)$.

3. To obtain a definite integral formula for $T_n(z)$, we set

$$t=z+\sqrt{z^2-1}.\ e^{i\phi} \text{ in §2}$$

so that

$$\frac{T_n(z)}{\sqrt{z^2-1}}=\frac{(n!)^2}{(2n)!\,\pi}\int_{-\pi}^{\pi}\left(\frac{t^2-1}{t-z}\right)^{(2n-1)/2}.\ (t-z)^{-\frac{1}{2}}d\phi$$

$$=\frac{(n!)^2}{(2n)!\,\pi}.\ 2^{(2n-1)/2}$$

$$\times\int_{-\pi}^{\pi}(z^2-1)^{-\frac{1}{4}}.\ e^{-i\phi/2}.\ (z+\sqrt{z^2-1}.\ \cos\phi)^{(2n-1)/2}d\phi$$

$$=\frac{(n!)^2}{(2n)!\,\pi}.\ 2^{(2n+1)/2}$$

$$\times\int_0^{\pi}(z^2-1)^{-\frac{1}{4}}.\cos\frac{\phi}{2}\,(z+\sqrt{z^2-1}.\cos\phi)^{(2n-1)/2}d\phi.$$

Consequently,

$$\frac{T_n(z)}{(z^2-1)^{\frac{1}{4}}}=\frac{2^{2n+1/2}.\ (n!)^2}{\pi.\ (2n)!}\int_0^{\pi}\cos\frac{\phi}{2}\,(z+\sqrt{z^2-1}\cos\phi)^{(2n-1)/2}d\phi.$$

Obviously this formula resembles Laplace's definite integral formula for $P_n(z)$.

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NORMED ALGEBRAS, ISOMORPHISM AND THE ASSOCIATIVE POSTULATE

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1. Notation, Definitions and Introduction.

Notation. The following symbols will stand for the entities mentioned against them :

R: the field of real numbers; C: the field of complex numbers.

Q: the field of quaternions.

λ, μ with or without suffixes: complex numbers.

$l, a, b, c, \alpha, \beta, \theta, \epsilon$ with or without suffixes: real numbers.

r, n, p, q, t: integers $\geqslant 0$; m any integer $\geqslant -1$.

$\bigcirc(\lambda, \alpha)$: circle in the complex plane with centre λ and radius α.

(complex or real) A: an Algebra which is a linear space (with complex or real numbers as the scalar multipliers).

z: the zero element of A.

u: the unit element of A, if such exists.

x, y, with or without suffixes : elements of A.

s with or without suffixes : scalar multipliers of A.

$P_n(\overleftarrow{x}), A_n(\overleftarrow{x})$: elements of A of the forms $\sum_{m=1}^{h} s_m \overset{\leftarrow m}{x}$ and $u \sum_{m=0}^{h} s_m \overset{\leftarrow n}{x}$ respectively, where $\overset{\rightarrow n}{x}$ and $\overset{\leftarrow n}{x}$ are defined by the formulae

$$\overset{\leftarrow n+1}{x} = x.\overset{\leftarrow n}{x}\ ;\ \overset{\rightarrow n+1}{x} = \overset{\rightarrow n}{x}.x \text{ for } n \geqslant 1, \text{ and } \overset{\rightarrow 1}{x} = \overset{\leftarrow 1}{x} = x, \text{ and } \overset{\leftarrow 0}{x} = u = \overset{\rightarrow 0}{x}. \quad (1)$$

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$$P(n, m, \overleftarrow{x}, \lambda) = \sum_{p=1}^{m} {}_{p+m-1}C_{p-1} \overleftarrow{x}^{p} \lambda^{-p},$$

where ${}_{\alpha}C_n$ is the coefficient of β^n in the expansion of $(1+\beta)^{\alpha}$ in powers of β.

Definitions. Normed A: an A such that to each element x of it corresponds a real number $\|x\|$ with the properties

$$\|x\| = 0 \text{ or } > 0 \text{ according as } x = z \text{ or } \neq z; \quad (2)$$

$$\|sx\| = |s| \, \|x\|; \quad (3)$$

$$\Big\| \sum_{p=1}^{m} x_p \Big\| \leqslant g \Big[\sum_{p=1}^{m} \|x_p\| \Big], \quad (4)$$

where $g(\alpha)$ is a monotonic increasing function of α, and tends to 0 as α tends to $+0$.

$$x_n \rightarrow x. \equiv \|x_n - x\| \rightarrow 0.$$

Null-sequence: any sequence (x_n) such that $x_n \rightarrow z$.

Left Topological divisor of z, (shortly $L.T.$): any element x such that for some non-null sequence (y_n), $x\,y_n \rightarrow z$.

Right Topological divisor of z (shortly $R.T$): any element x such that for some non-null sequence (y_n), $y_n x \rightarrow z$.

Two-sided Topological divisor of z (shortly T): an element x which is both an $L.T.$ and an $R.T.$ (z is one such).

Left continuity $\equiv y_n \rightarrow z$ implies $y_n x \rightarrow z$.

Right continuity $\equiv y_n \rightarrow z$ implies $x y_n \rightarrow z$.

Association $\overleftarrow{A}_m \equiv \overleftarrow{x}^{p} . \overleftarrow{x}^{n} = \overleftarrow{x}^{p+n}$ for every x, and for $1 \leqslant p \leqslant m$.

Association $\overrightarrow{A}_m \equiv \overrightarrow{x}^{n} . \overrightarrow{x}^{p} = \overrightarrow{x}^{n+p}$ for every x, and for $1 \leqslant p \leqslant m$.

Association $\overleftarrow{I}(a, b, c) \equiv$ If $y(ax^2+bx+c) = u$

then $$y[(ax^2+bx+c)\overleftarrow{x}^{n}] = \overleftarrow{x}^{n}.$$

Association $\overrightarrow{I}(a, b, c) \equiv$ If $(ax^2+bx+c)\,y = u$

then $$[\overrightarrow{x}^{n}(ax^2+bx+c)]\,y = \overrightarrow{x}^{n}.$$

Association $\overset{*}{I} \equiv (x_1x_2)x_3 = x_1(x_2x_3)$ if two of x_1, x_2, x_3 are equal roots of $x^2 = x \neq z$;

(any such x is called an idem-potent).

Association $\overset{*}{A} \equiv (x_1x_2)x_3 = x_1(x_2x_3)$ if x_1, x_2, x_3 all satisfy the equation $(x^2)^2 = -x^2$.

Left-inversive A : an A in which to each $x \neq z$ corresponds at least one y with the property $yx = u$.

Right-inversive A: an A in which to each $x \neq z$ corresponds at least one y with the property $xy = u$.

Indefinite-inversive A: an A in which to each $x \neq z$ corresponds at least one y such that at least one of the relations $xy = u$ and $yx = u$ holds.

Introduction. This paper presents improvements on known results (as outlined in a recent survey by Kaplansky [2]) which have sought to obtain conditions for the isomorphism of an A to one of R, C, and Q. Completeness is nowhere assumed, and the usual norm conditions of the previous authors (viz., $\|x+y\| \leqslant \|x\| + \|y\|$; $\|xy\| \leqslant \|x\| \|y\|$) are replaced by (4) and one sided continuity or separate continuities (the latter is needed only in dealing with indefinite inversion or absence of T's); and the minimum degree of association (in multiplication) necessary to secure the results has been ascertained. It will be noted that either $\overset{*}{I}$ or $\overset{*}{A}$ is needed only for a non-commutative real A and that the theorems on inverses do not assume any of the inverses to be two sided or unique. To facilitate reference the results are presented together below.

LEMMA I. *In the presence of right continuity, to each x in a normed A correspond c_x and l_x such that*

$$\|xy\| \leqslant c_x \|y\| \, ; \; l_x = \lim_{n \to \infty} \sup \|\overset{\leftarrow n}{x}\|^{1/n} \leqslant c_x. \qquad (5)$$

(The proof of this is implied in that of Theorem I in Gelfand [1]. In short, the lemma is obvious if $x = z$, and if $x \neq z$, sup $\|xy\|$ for $\|y\| = 1$ is finite on account of right continuity.)

THEOREM 1 (*a*). *In the presence of* $\overleftarrow{A}_2$ *and right continuity, to each* $x \neq z$ *of a normed complex A corresponds a* λ_x *and a sequence* $P_n(\overleftarrow{x})$ *such that*

$$|\lambda_x| = l_x \tag{6}$$

$$\| P_n(\overleftarrow{x}) \| = 1 \tag{7}$$

$$(x\lambda_x - x^2)\, P_n(\overleftarrow{x}) \to z. \tag{8}$$

THEOREM 1 (*b*). *In the presence of unit element, and right continuity, to each x of a normed complex A correspond* λ_x *and a sequence* $A_n(\overleftarrow{x})$ *such that*

$$|\lambda_x| = l_x \tag{9}$$

$$\| A_n(\overleftarrow{x}) \| = 1 \tag{10}$$

$$(u\lambda_x - x) A_n(\overleftarrow{x}) \to z. \tag{11}$$

THEOREM 2. *A normed complex A with more than one element is isomorphic to C*

(*i*) *in the presence of* $\overleftarrow{A}_2$ *and right continuity and in the absence of non-zero L.T's*;

or (*ii*) *in the presence of* $\overleftarrow{A}_4$, $\overrightarrow{A}_4$, *and both continuities and in the absence of T's*;

or (*iii*) *in the presence of unit element and right continuity and in the absence of non-zero L.T's*;

or (*iv*) *in the presence of* $\overleftarrow{A}_2$, $\overrightarrow{A}_2$ *and both continuities and unit element and in the absence of non-zero T's*;

or (*v*) *in the presence of left inversion, right continuity and* $\overleftarrow{I}$ (0, 1, 0).

or (*vi*) *in the presence of indefinite inversion, both continuities and* $\overleftarrow{I}$ (0, 1, 0) *and* $\overrightarrow{I}$ (0, 1, 0).

THEOREM 3. *A normed real commutative A with more than one element is isomorphic to one of R and C*

(*i*) *in the presence of* $\overleftarrow{A}_4$ *and right continuity and in the absence of non-zero L.T's*;

or (*ii*) *in the presence of* $\overleftarrow{A}_8$ *and* $\overrightarrow{A}_8$ *and both continuities, and in the absence of non-zero* T'*s*;

or (*iii*) *in the presence of a unit element and* $\overleftarrow{A}_2$, *and right continuity and in the absence of non-zero* $L.T$'*s*;

or (*iv*) *in the presence of a unit element and* $\overleftarrow{A}_4$ *and* $\overrightarrow{A}_4$ *and both continuities and in the absence of non-zero* T'*s*;

or (*v*) *in the presence of left inversion and* $\overleftarrow{A}_2$ *and right continuity and* $\overleftarrow{I}$ (a, b, c);

or (*vi*) *in the presence of indefinite inversion and* $\overleftarrow{A}_2$ *and both continuities and* $\overleftarrow{I}$ (a, b, c) *and* $\overrightarrow{I}$ (a, b, c).

THEOREM 4. *A real normed non-commutative A is isomorphic to Q*

(*i*) *in the presence of* $\overleftarrow{A}_4$, $\overset{*}{I}$, $\overset{*}{A}$ *and right continuity and in the absence of non-zero* $L.T$'*s*;

or (*ii*) *in the presence of* $\overleftarrow{A}_8$. *and* $\overrightarrow{A}_8$, $\overset{*}{I}$, $\overset{*}{A}$, *and both continuities and in the absence of non-zero* T'*s*;

or (*iii*) *in the presence of a unit element*, $\overleftarrow{A}_2$, $\overset{*}{A}$ *and right continuity and in the absence of non-zero* $L.T$'*s*;

or (*iv*) *in the presence of a unit element*, $\overleftarrow{A}_4$, $\overrightarrow{A}_4$, $\overset{*}{A}$, *both continuities and in the absence of non-zero* T'*s*;

or (*v*) *in the presence of left inversion*, $\overleftarrow{A}_2$ *and* $\overset{*}{A}$, *and* $\overset{*}{I}$ (a, b, c) *and right continuity*;

or (*vi*) *in the presence of indefinite inversion* $\overleftarrow{A}_2$, $\overset{*}{A}$ *and* $\overleftarrow{I}(a, b, c)$ *and* $\overrightarrow{I}$ (a, b, c), *and both continuities.*

2. **Functions of a Complex Variable.**

We shall be concerned with functions of the type $\phi(\lambda) = \sum_{m=1}^{p} x_m f_m(\lambda)$, where the $f_m(\lambda)$ are ordinary analytic functions of λ or functions with a finite number of

ordinary discontinuities. It is obvious that the notions of contour integration are applicable to such functions and that Taylor's theorem and Cauchy's theorem are valid for such analytic functions. In particular we have

$$\left.\begin{aligned}
&\int_{\Gamma}\varphi(\lambda)d\lambda = \sum_{m=1}^{p} x_m\int_{\Gamma} f_m(\lambda)d\lambda\,;\\
&\int_{\Gamma_1}\varphi(\lambda)d\lambda+\int_{\Gamma_2}\varphi(\lambda)d\lambda = \int_{\Gamma_1+\Gamma_2}\varphi(\lambda)d\lambda\,:\\
&\left\|\int_{\Gamma}\varphi(\lambda)d\lambda\right\| \leqslant g\left[\int\|\varphi(\lambda)\|\,|d\lambda|+\epsilon\right] \text{ for } \epsilon>0\,;\\
&\int_{\Gamma}[\varphi_1(\lambda)+\varphi_2(\lambda)]\,d\lambda = \int_{\Gamma}\varphi_1(\lambda)d\lambda+\int_{\Gamma}\varphi_2(\varphi)d\lambda.
\end{aligned}\right\} \tag{12}$$

In addition to these results we shall need the following

LEMMA 2. *Let* $0<\beta_1<\beta\,;\, 0<\beta_2<\beta$. *Let* $\varphi_n(\lambda)$ *be a sequence of functions of the above type analytic inside and on* $\bigcirc(\mu_1, \beta) = \Gamma$ *and uniformly bounded on the circumference; and let* $\varphi_n(\lambda)$ *tend uniformly to* z *on* $\bigcirc(\mu_1, \beta_1) = \Gamma_1$. *Then* $\varphi_n(\lambda)$ *tends uniformly to* z *inside and on* $\bigcirc(\mu_1, \beta_2)$.

PROOF. Let $|\mu-\mu_1|\leqslant\beta_2$. Then, by Taylor's theorem

$$\phi_n(\mu) = \sum_{m=0}^{p} y_{n,m}(\mu-\mu_1)^m + R(p, n, \mu), \tag{13}$$

where

$$y_{n,m} = \frac{1}{2\pi i}\int_{\Gamma_1}\varphi_n(\lambda)/(\lambda-\mu_1)^{m+1}d\lambda, \tag{14}$$

$$R(p, n, \mu) = \frac{1}{2\pi i}\int_{\Gamma}\varphi_n(\lambda)\,(\mu-\mu_1)^{p+1}/[(\lambda-\mu_1)^{p+1}(\lambda-\mu)]d\lambda. \tag{15}$$

Let now $\epsilon>0$. Then by (15), the boundedness hypothesis and (12), a q exists such that

$$\|R(q, n, \mu)\| < \epsilon/2 \text{ for every } n \text{ and for } |\mu-\mu_1|\leqslant\beta_2. \tag{16}$$

For this q there corresponds by (14), (4), (12) and the hypothesis about Γ_1 an integer $n(\epsilon)$ such that

$$\left\|\sum_{m=0}^{q} y_{n,m}(\mu-\mu_1)^m\right\| \leqslant g\left[\sum_{m=0}^{q}\|y_{n,m}(\mu-\mu_1)^m\|\right] < \frac{\varepsilon}{2} \text{ for } n\geqslant n(\varepsilon), \text{ and } |\mu-\mu_1|\leqslant\beta_2. \tag{17}$$

The lemma now follows from (16), (17) and (4).

3. Proofs of Theorems I (a) and I (b).

PROOF OF THEOREM I (*a*). Let $x \neq z$. Then the l_x of Lemma 1 is either 0 or > 0.

Case (*i*) $l_x = 0$ Let $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$. be a sequence of positive numbers with limit 0 Then to n corresponds an integer m_n such that

$$\| (\overleftarrow{x}^{2+m_n}) / \varepsilon_n^{2+m_n} \| < 1, \text{ since } l_x = 0. \tag{18}$$

Now let $b_n = \sup\limits_{|\lambda| = \varepsilon_n} \| P(m_n, 0, \overleftarrow{x}, \lambda) \|$, and $\Gamma_n = \bigcirc(0, \varepsilon_n)$.

Then by (12)

$$\| x \| = \| \frac{1}{2\pi i} \int_{\Gamma_n} P(m_n, 0, \overleftarrow{x}, \lambda)\, d\lambda \| \leqslant g\,(b_n \varepsilon_n + \varepsilon)$$

for every $\varepsilon > 0$. (19)

Hence (b_n) is a divergent sequence, and there exists a set of numbers λ_n such that

$$|\lambda_n| = \varepsilon_n \,;\; \| P(m_n, 0, \overleftarrow{x}, \lambda_n) \| > \tfrac{1}{2}\, b_n \to \infty \text{ as } n \to \infty. \tag{20}$$

We now note, in virtue of $\overleftarrow{A}_2$, the identity

$$(x\lambda - x^2)\, P(n, 0, \overleftarrow{x}, \lambda) = x^2 - (\overleftarrow{x}^{n+2}) / \lambda^n. \tag{21}$$

We now take $\lambda_1 = 0\ (= l_x)$;

$$P_n(\overleftarrow{x}) = P(m_n, 0, \overleftarrow{x}, \lambda_n) / \| P(m_n, 0, \overleftarrow{x}, \lambda_n \|.$$

Then (6) and (7) of the theorem are satisfied and (8) is satisfied in virtue of (21), (20), (18), Lemma 1, and (4).

Case (*ii*) $l_x > 0$. Without loss of generality we may take $l_x = 1$. We now note, in virtue of $\overleftarrow{A}_2$, the identity

$$(\lambda x - x^2)\, P(n, m, \overleftarrow{x}, \lambda) = \lambda x\, P(n, m\ \ 1, \overleftarrow{x}, \lambda) -$$
$${}_{n+m-1}C_{n-1}\, \overleftarrow{x}^{n+2} / \lambda^n,\ m \geqslant 0,\ n \geqslant 1. \tag{22}$$

We now observe that

$\overline{\lim\limits_{n\to\infty}} \| P(n, m, x, \lambda) \| > 0$, for m fixed, λ fixed, and $|\lambda| > 1$. (23)

For, otherwise, (22) gives the existence of a sequence n_r such that $x P(n_r, m-1, \overleftarrow{x}, \lambda) \to z$ as $r \to \infty$, and multiplying this on the left by $\lambda x - x^2$ and repeating the process m times, it would follow from right continuity that $\overleftarrow{x}^{m+2} = z$, contrary to supposition that $l_\tau = 1$.

Hence to each $m \geqslant 0$ and each λ with $|\lambda| > 1$ correspond a pair of positive numbers $b(m, \lambda)$ and $a(m, \lambda)$ such that

$$a(m, \lambda) > m\,;$$

$$\left.\begin{array}{l} \| P(n, m, \overleftarrow{x}, \lambda) \| > b(m, \lambda) > 0\,; \\ \| {}_{n+m-1}C_{n-1} \overleftarrow{x}^{n+2} / \lambda^n \| < (|\lambda| - 1) b(m, \lambda) \end{array}\right\} \text{ for } n > a(m, \lambda)\,. \tag{24}$$

Now let t be any integer $> 28\pi$, and $\alpha_t = 2(\sin \pi/t)/(1-4 \sin \pi/t)$ and consider a regular polygon of t sides in the complex plane with centre 0 and radius $1+2\alpha_t$. Let $\lambda_1 = (\lambda_{t+1}), \lambda_2, \ldots \lambda_t$ be its vertices with amplitudes $\theta_1 (= \theta_{t+1}), \theta_2 \ldots \theta_t$ respectively.

Now for any $m \geqslant 0$ let $n_{m,t} = \max\limits_{0 \leqslant q \leqslant m, 1 \leqslant p \leqslant t} a(q, \lambda_p)$, and

$$\phi(p, m, \lambda) = \sum_{q=0}^{m} P(n_{m,t}, q, \overleftarrow{x}, \lambda_p)\ (\lambda_p - \lambda)^q \lambda_p^{-q} \quad \text{for } 1 \leqslant p \leqslant t.$$

Now the set of numbers $\|\varphi(p, m, \lambda)\|$ for $1 \leqslant p \leqslant t$, $|\lambda - \lambda_p| \leqslant 12\,\alpha_t$ must be unbounded. For, otherwise, in the common region of $\bigcirc(\lambda_p, 12\,\alpha_t)$ and $\bigcirc(\lambda_{p+1}, 12\,\alpha_t)$ the infinite sequence $\varphi(p+1, m, \lambda) - \varphi(p, m, \lambda)$ would be bounded, while as m tends to infinity it would tend to z uniformly in the common region of $\bigcirc(\lambda_p, \alpha_t)$ and $\bigcirc(\lambda_{p+1}, \alpha_t)$; and by taking $\frac{1}{2}(\lambda_p + \lambda_{p+1})$, $\frac{1}{2}\alpha_t$, $11\,\alpha_t$, and $12\,\alpha_t$ for the numbers μ_1, β_1, β_2 and β of Lemma 2, it will follow that the difference would tend to z uniformly on S_p, the straight line segment from

$$\tfrac{1}{2}(\lambda_p + \lambda_{p+1}) - 11\,\alpha_t \exp\left\{ \frac{i}{2}(\theta_p + \theta_{p+1}) \right\}$$

to

$$\tfrac{1}{2}(\lambda_p + \lambda_{p+1}) + \tfrac{1}{2}\alpha_t \exp\left\{ \frac{i}{2}(\theta_p + \theta_{p+1}) \right\}.$$

Now consider the contour Γ_p consisting of S_p, S_{p+1}, and the two small circular arcs centred at 0 connecting their ends; and observe that

$$\int_{\Gamma_p} \varphi(p, m, \lambda)\lambda^{n-1} d\lambda = z \tag{25}$$

and that

$$\| \varphi(p, m \ \lambda) - P(m, 0, \overleftarrow{x}, \lambda) \| \to 0$$

uniformly in λ, on the outer arc of Γ_p as $m \to \infty$. (26)

Now summing the t integrals of the kind (25) for any fixed m and making $m \to \infty$ with due attention to (26) and (12), it would follow on account of the assumed boundedness of the $\varphi(p, m, \lambda)$ that

$$\| \overleftarrow{x}^n \| = (1 - 7\alpha_t)^n . \ O\ (1),$$

and hence that

$$\limsup_{n\to\infty} \| \overleftarrow{x}^n \|^{1/n} \leqslant 1 - 7\,\alpha_t < 1$$

contrary to the supposition that $l_r = 1$.

Hence the above set of numbers is unbounded and provides the existence of λ_t, m_t and n_t such that

$$\left.\begin{array}{l} |\lambda_t| = 1 + 2\,\alpha_t, \\ n_t > a\,(m_t, \lambda_t);\ n_t > a\,[(m_t - 1), \lambda_t]; \\ \| P(n_t, m_t, \overleftarrow{x}, t) \| \geqslant (13\,\alpha_t)^{-1} \| P(n_t, m_t - 1, \overleftarrow{x}, \lambda_t) \| > 0. \end{array}\right\} \tag{27}$$

We now write

$$P_t(\overleftarrow{x}) = P(n_t, m_t, \overleftarrow{x}, \lambda_t) / \| P(n_t, m_t, \overleftarrow{x}, \lambda_t) \| \tag{28}$$

and observe that in virtue of (27), (23), (24) and (4)

$$\| (\lambda_t x - x^2)\ P_t(\overleftarrow{x}) \| \leqslant g\,[13\alpha_t + 2\,\alpha_t] = g\,[15\alpha_t] \to 0 \text{ as } t \to \infty. \tag{29}$$

The set of numbers λ_t has limit points forming a closed set on $|\lambda| = 1$. If we now take λ_r to be any one of these, (6), (7) and (8) of the theorem are verified in virtue of (29), (28), Lemma 1 and (4) for a subsequence of (λ_t) with λ_r as its limit.

Proof of Theorem 1 (*b*). The proof follows the lines of the above with $\lambda u - x$ in place of $\lambda x - x^2$ and $\lambda^{-1}\{u + P(n, m, \overleftarrow{x}, \lambda)\}$ in place of $P(n, m, \overleftarrow{x}, \lambda)$ and y/λ in place of

$P(n, -1, \overleftarrow{x}, \lambda)$. It is obvious that the associative postulate for multiplication is not needed here.

Note about dual theorems. We observe that the proof of any of the theorems of this paper carries along with it the proof of the dual theorem obtained by interchange of arrows and simultaneous interchange of the continuities.

4. Proof of Theorem 2.

Cases (i) *and* (ii):—Let $x \neq z$.

Then in case (*i*) $x^2 = x\lambda_x$ by Theorem 1 (*a*);

and in case (*ii*) by $\overleftarrow{A}_4$ and $\overrightarrow{A}_4$ and absence of T's and Theorem 1 (*a*) and its dual, and Lemma 1 and its dual, follows

$$(x^2 - x\lambda_x)\ (x^2 - x\lambda_x') = (x^2 - x\lambda_x')\ (x^2 - x\lambda_x) = z,$$

where $\|\lambda_x'\| = \limsup_{n\to\infty} \|\overleftarrow{x^n}\|^{1/n}$. From this again, on account of absence of T's follows either $x^2 = \lambda_x x$ or $x^2 = x\lambda'_x$. In either case therefore we have a μ_x such that

$$x^2 = x\,\mu_x, \tag{30}$$

where on account of absence of T's, and the hypothesis that $x \neq z$,

$$|\mu_x| > 0 \tag{31}$$

so that

$$x/\mu_x = e_x, \text{ say, is an idempotent.} \tag{32}$$

If now e_x and e_y are two idempotents, then either

$$e_x = e_y; \tag{33}$$

or

$$e_x - e_y \neq z; \tag{34}$$

whence by (30) and (31) numbers λ_1 and λ_2 would exist such that

$$\left.\begin{aligned}(e_x - e_y)^2 &= \lambda_1\,(e_x - e_y),\ \lambda_1 \neq 0, \text{ and}\\ (e_x + e_y)^2 &= \lambda_2\,(e_x + e_y)\end{aligned}\right\} \tag{35}$$

whence by addition, would follow

$$\mu_1 e_x = \mu_2 e_y, \text{ where } \mu_1 \neq \mu_2 \tag{36}$$

which is a contradition since e_x and e_y are idempotents; hence

$$e_x = e_y \tag{37}$$

and there exists a u, such that $u^2 = u \neq z$, and $x = \mu_1 u$ and the desired result follows.

Cases (iii), (iv), *and* (v) Obvious by Theorem 1 (*b*) and its dual.

Case (vi). For sufficiently large α, we have by Lemma 1 and its dual

$$\left.\begin{array}{l} (\alpha u - x) \left[\sum_{m=0}^{n} (\overleftarrow{x}^{m})/\alpha^{m+1}\right] \to u \text{ as } n \to \infty; \text{ and} \\ \left[\sum_{m=0}^{n} (\overrightarrow{x}^{m})/\alpha^{m+1}\right] (\alpha u - x) \to u \text{ as } n \to \infty. \end{array}\right\} \tag{38}$$

If now y is a left inverse of $\alpha u - x$ it follows by $\overleftarrow{I}(0, 1, 0)$, (38), and right continuity that

$$y \left[(\alpha u - x) \sum_{m=0}^{n} \left\{ (\overleftarrow{x}^{m})/\alpha^{m+1} \right\}\right] = \sum_{m=0}^{n} \left[(\overleftarrow{x}^{m})/\alpha^{m+1}\right] \to y \text{ as } n \to \infty. \tag{39}$$

Hence by right continuity and (38), y is a right inverse of $(\alpha u - x)$; and by a dual argument follows

$$\sum_{m=0}^{n} \left[(\overrightarrow{x}^{m})/\alpha^{m+1}\right] \to y \text{ as } n \to \infty. \tag{40}$$

The same conclusions (39) and (40) follow if y is a right inverse of $\alpha u - x$ From (39) and (40), by subtraction it follows that

$$\sum_{m=0}^{\infty} \left[(\overleftarrow{x}^{m} - \overrightarrow{x}^{m})/\alpha^{m+1}\right] = z \text{ for all sufficiently large } \alpha. \tag{41}$$

Hence by (4) and an obvious limit process follows

$$\overrightarrow{x}^{m} = \overleftarrow{x}^{m} \text{ for every } x. \tag{42}$$

Now from Theorem 1 (*b*), $\overleftarrow{I}$ (0, 1, 0) and $\overrightarrow{I}$ (0, 1, 0) it follows that $x - \lambda_x u$ of Theorem 1 (*b*) cannot have any inverse. Hence the desired result.

5. **Lemma on Real Normed A.** Given a real normed A, we consider the set of ordered pairs (x, y), denoted for shortness by $(x+iy)$, subject to the following rules:

$$(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i\ (y_1+y_2)$$

$$(x_1+iy_1)\ (x_2+iy_2) = (x_1 x_2-y_1 y_2)+i\ (x_1\ y_2+y_1\ x_2)$$

$$(a+ib)\ (x+iy) = (ax-by)+i\ (ay+bx)$$

$$(x+iy)\ (a+ib) = (xa-yb)+i\ (xb+ya)$$

$$\| x+iy \| = \sup_{0 \leqslant \theta \leqslant 2\pi} [\| x\cos\theta - y\sin\theta \| + \| x\sin\theta + y\cos\theta \|].$$

It is obvious that this set is a normed complex A with $2\,g(\alpha)$ in place of $g(\alpha)$ in (4), with $z+iz$ for its zero element and with $u+iz$ for its unit element in case A has an unit element. It is also obvious that continuity of either kind in the given real A implies the corresponding continuity in this complex A; and also that if (x_n) and (y_n) are null sequences in real A, then (x_n+iy_n) is a null sequence in the complex A and conversely; and that if $\overleftarrow{A}_m$ or $\overrightarrow{A}_m$ exists in A, it exists also in the complex A for each of the elements $x+iz$. Applying therefore Theorems 1 (a) and 1 (b) to the element $x+iz$ of this complex A we obtain the following two lemmas.

LEMMA 3. *In the presence of* $\overleftarrow{A}_4$ *and right continuity, to each* $x \neq z$ *of a real normed* A, *correspond a pair of real numbers* a_x *and* b_x *and a sequence* $\overleftarrow{P_n}(x)$ *such that*

$$a_x^2+b_x^2 = l_x^2 \tag{43}$$

$$\overleftarrow{P_n}(x) \text{ is not a null sequence} \tag{44}$$

$$[(x^2-x\,a_x)^2+x^2\,b_x^2]\ \overleftarrow{P_n}(x) \to z \text{ as } n \to \infty. \tag{45}$$

LEMMA 4. *In the presence of* $\overleftarrow{A}_2$, *right continuity, and a unit element* u, *to each* x *of a normed real* A *correspond real numbers* a_x *and* b_x *and a sequence* $\overleftarrow{A_n}(x)$ *such that*

$$a_x^2+b_x^2=l_x^2; \tag{46}$$

$$A_n(x) \text{ is not a null sequence}; \tag{47}$$

$$\{(x-u\,a_x)+u\,b_x^2\}\,\overleftarrow{A_n(x)} \to z \text{ as } n\to\infty. \tag{48}$$

From these two lemmas we obtain the following two lemmas.

Lemma 5. *In cases (i) and (ii) of each of Theorems 3 and 4, to each $x \neq z$ correspond an idempotent e_x and a pair of real numbers a_x, b_x such that*

$$(x^2-2x\,a_x)^2+x^2\,b_x^2=z,\ a_x^2+b_x^2\neq 0$$

$$(\textit{viz. at least one of } a_x \textit{ and } b_x\neq 0); \tag{49}$$

$$x=x\,e_x=e_x\,x; \tag{50}$$

$$(x-a_x\,e_x)^2+b_x^2\,e_x=z. \tag{51}$$

Proof. The first part of (49) follows from the hypotheses and Lemma 3 and its dual as (30) did in cases (i) and (ii) of Theorem 2. The second part of (49) follows from the absence of non-zero T's. From (49) it follows by $\overleftarrow{A_4}$ and absence of T's that

$$x(x^2-2\,x\,a_x)+(a_x^2+b_x^2)\,x=(x^2-2\,x\,a_x)\,x+(a_x^2+b_x^2)\,x=z.$$

We now write

$$e_x=-(x^2-2\,xa_x)/(a_x^2+b_x^2) \tag{52}$$

and it is verified that (50) is true, and hence $e_x\neq z$, and is an idempotent. From (52) and (50) follows (51).

Lemma 6. *In cases (iii) and (iv) of each of Theorems 3 and 4 to each $x\neq z$ correspond a pair of real numbers a_x and b_x such that*

$$(x-u\,a_x)^2+b_x^2\,u=z. \tag{53}$$

Proof. Obvious from Lemma (4), its dual, and the absence of T's.

Lemma 7. *In cases (i) and (ii) of Theorem 3, any two idempotents (which exist by Lemma 5) are equal.*

Proof. Let e_1 and e_2 be two idempotents. We observe that A is commutative, so that $e_1\,e_2=e_2\,e_1$, and

$$(e_1+e_2)\ (e_1-e_2) = (e_1-e_2)\ (e_1+e_2) = e_1-e_2. \tag{54}$$

Now either

$$e_1-e_2 = z; \tag{55}$$

or

$$e_1-e_2 \neq z. \tag{56}$$

If (56) were true, we should have by Lemma 3, $\overleftarrow{A_4}$ and absence of T's,

$$(e_1-e_2)^3-2\,\alpha_1(e_1-e_2)^2+(\alpha_1^2+\beta_1^2)\ (e_1-e_2) = z,\ \alpha_1^2+\beta_1^2>0 \tag{57}$$

and hence by (54), absence of T's, (56), and $\overleftarrow{A_4}$

$$(e_1-e_2)^2-2\,\alpha_1\,(e_1-e_2)+(\alpha_1^2+\beta_1^2)\ (e_1+e_2) = z,$$

whence

$$e_1\,e_2 = l_1\,e_1+l_2\,e_2 \tag{58}$$

where either (i) $l_1 \neq l_2$, or (ii) $l_1 = l_2 > \frac{1}{2}$ by (57). (59)

Now, $e_1+e_2 \neq z$ [otherwise $e_1 = e_1^2 = e_2^2 = e_2 = \frac{1}{2}\ (e_1+e_2) = z$, a contradiction] and hence, as in (57)

$$(e_1+e_2)^3-2\,a\,(e_1+e_2)^2+(a^2+b^2)\ (e_1+e_2) = z;\ (a^2+b^2)>0; \tag{59a}$$

and hence by (58)

$$e_1\,[1+4\,l_1+2\,l_1^2+2\,l_1\,l_2-2a\,(1+2\,l_1)+a^2+b^2]$$
$$+e_2\,[1+4\,l_2+2\,l_2^2+2\,l_1\,l_2-2a\,(1+2\,l_2)+a^2+b^2] = z;$$

now, if either of the coefficients of e_1 and e_2 in this were different from zero, it would follow that $e_1 = e_2$ [contrary to (56)]; and hence that both should be equal to 0, and hence by subtraction and addition we obtain in case $l_1 \neq l_2$

$$l_1+l_2 = -2\ (1-a)$$
$$(1+l_1+l_2-a)^2+b^2 = 0$$

and hence

$$a = 1,\ b = 0$$

and hence by (59a) and $\overleftarrow{A_4}$

$$[(e_1+e_2)^2-(e_1+e_2)]^2 = z$$

and hence by absence of T's $e_1\,e_2 = z$ and hence by (58), would follow

$$e_1 = e_2 ; \tag{60}$$

and from this contradiction of (56) would follow, via (59), $e_1 e_2 = \alpha(e_1+e_2)$, $\alpha > \frac{1}{2}$; whence $(e_1+e_2)^2 = (1+2\alpha)(e_1+e_2)$; (61)

and writing $c = 1+2\alpha$ it would follow that $(e_1+e_2)/c$ is an idempotent that commutes with e_1. Replacing, therefore, e_2 by this in the above argument from (56) to (61) it would follow that [either $e_1 = (e_1+e_2)/c$, which is false since $c > 2$ and e_1 and e_2 are idempotents, or]

$$e_1.(e_1+e_2)/c = \beta[e_1+(e_1+e_2)/c],\ \beta > \tfrac{1}{2},\ c = 1+2\alpha,\ \alpha > \tfrac{1}{2} ;$$

$$\text{viz. } e_1 + \alpha(e_1+e_2) = \beta[e_1(1+2\alpha)+e_1+e_2] ;$$

$$\text{viz. } e_1(1+\alpha)(1-2\beta) = (\beta-\alpha)e_2$$

and hence that $\beta = \alpha = \frac{1}{2}$, since otherwise we should have $e_1 = e_2$.

This is a contradiction, arising via (59), of (56), and taken along with (55) proves the lemma.

Lemma 8. *In cases* (*i*) *and* (*ii*) *of Theorem* 4, *any two idempotents* (*which exist by Lemma* 5) *are equal.*

Proof. Let e_1 and e_2 be any two idempotents, and let $y = e_2 \quad e_1 e_2 - (e_2 - e_1 e_2) e_1$. Then by $\overset{*}{I}$, $y e_1 = e_1 y = z$.

Hence by absence of T's

$$e_2 - e_1 e_2 - (e_2 - e_1 e_2) e_1 = z ; \text{viz. } (e_2-e_1) e_2 - (e_2-e_1) e_2 e_1 = z. \tag{62}$$

Hence by $\overset{*}{I}$

$$(e_2 - e_1)(e_2 - e_2 e_1) = z. \tag{63}$$

Also by $\overset{*}{I}$

$$(e_2 - e_2 e_1) e_1 = z. \tag{64}$$

From (63), (64) and absence of T's follows

either, $\quad e_1 = e_2$
or, $\quad e_2 - e_2 e_1 = z.$ } (65)

Similarly we have

either, $\quad e_1 = e_2$
or, $\quad e_1 - e_1 e_2 = z.$ } (66)

From (65) and (66) follows, by addition, either $e_1 = e_2$, or $(e_1 - e_2)^2 = z$ In the latter alternative follows $e_1 = e_2$ by absence of T's.

6. **Proof of Theorem 3.** In case (*vi*), we argue as in case (*vi*) of Theorem 2 and then apply Lemma 4.

In cases (*i*) and (*ii*) we apply Lemma 4, its dual, and Lemmas 5 and 7.

In the other cases we appeal to Lemma 4, its dual, and Lemma 6.

In all cases, a unit element u exists and to each x correspond a pair of real numbers a_x and b_x such that

$$(x - a_x u)^2 + b_x^2 u = z. \tag{67}$$

If $b_x = 0$ it follows in all cases that $x = a_x u$. If for some x, $b_x \neq 0$, then there exists in A an element, i, such that $i^2 + u = z$. Let j be any other such element. Then, by substituting $i+j$, $i-j$ for x in (67) it follows that either,

$$i = \pm j \text{ (in case one at least of } a_{i+j}, \text{ and } a_{i-j} \text{ is} \neq 0), \tag{68}$$

$$\text{or, } ij + ji = 2\,\alpha\, u, \text{ where } |\alpha| < 1 \text{ (in case } a_{i+j} = a_{i-j} = 0). \tag{69}$$

[We observe that the argument from (67) to (69) is independent of commutation in multiplication]. In the latter case we have by commutativity of i and j

$$[i + (\alpha - 1)\, j]\; j = j\; [i + (\alpha - 1)\, j] = u$$

and hence by either $\overleftarrow{I}$ or $\overrightarrow{I}$ or absence of T's follows $i + \alpha j = z$ and hence $|\alpha| = 1$, contrary to (69).

Hence, in all cases

$$\left.\begin{array}{l} x = a_x u \text{ if } b_x = 0\,; \\ x = a_x u + b'_x\, i, \text{ where } b'_x = \pm b_x \text{ if } b_x \neq 0. \end{array}\right\} \tag{70}$$

Hence, if every $b_x = 0$, A is isomorphic to R; otherwise it is isomorphic to C, since $a_1 u + b_1 i = a_2 u + b_2 i$ if and only if $a_1 = a_2$, and $b_1 = b_2$.

7. **Proof of Theorem 4.** Except for the use of Lemma 8 in place of Lemma 7 in cases (*i*) and (*ii*), the proof follows the lines of that of Theorem 2 until we reach (67). The proof of Theorem 3 then shows that there exist at least two elements i and j such that

$$i^2 = j^2 = -u\,;\ ij \neq ji \tag{71}$$

for, otherwise A would be commutative.

Now (68) and (69) give

$$(i+\alpha j)\,j + j\,(i+\alpha j) = z,$$

whence

$$\left(\frac{i+\alpha j}{\sqrt{1-\alpha^2}}\right)^2 + u = z$$

and (71) can be replaced by

$$i^2 = j^2 = -u,\ ij + ji = z,\ ij \neq ji,\ ij \neq z. \tag{72}$$

Also by $\overset{*}{A}$, $ij \pm i \neq z$ in (72) and hence using (67) with ij, and $ij \pm i$ in place of x it follows by (72) and $\overset{*}{A}$ that

$$(ij)^2 + b^2 u = z,\ b \neq 0 \tag{73}$$

and hence by (72) and $\overset{*}{A}$ that

$$(ij)^2 + u = z. \tag{74}$$

It is well known that (67), (72), (74) and $\overset{*}{A}$ imply the desired isomorphism to Q.

Addendum (April 3, 1950). The referee has brought to my notice that in a paper entitled "On the power-associativity of rings" in *Summa Brasiliensis Mathematicae*, Vol. II (1948) Fasc. 2, pp. 21-32, A. Albert has proved that the postulates $\overrightarrow{A}_m$ and $\overleftarrow{A}_m$ of the text are all equivalent for $m > 4$. In view of this, one needs only $\bar{A}_5$ by way of hypothesis, instead of $\overrightarrow{A}_8$ and $\overleftarrow{A}_8$ in Theorem 3 (*ii*) and in Theorem 4 (*ii*) above.

The condition on $g(\alpha)$ in (4) may be omitted. For, writing $S = \sum_1^n \| x_t \|$, it follows from

$$\| \sum_1^n x_t \| \leqslant g \, [\sum_1^n \| x_t \|] \tag{4 a}$$

in case at least one $\| x_t \|$ in $S \neq 0$, that

$$\| \sum_1^n x_t \| \leqslant S \| \sum_1^n (x_t/s) \| \leqslant g(1) \sum_1^n \| x_t \|, \text{ (using (3))}, \tag{4 b}$$

which is obviously true if every $x_t = z$, and therefore equivalent to (4) with $g(\alpha)$ replaced by $g(1)\,\alpha$. It is obvious that $g(1) \geqslant 1$.

Note added August 12, 1950. On a perusal of Albert's paper mentioned above, which was sent to me by the editor, it appears that the results of this paper admit of further improvements in respect of the Associative postulate. These will appear shortly.

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A CALCULUSF OR FRE'CHET VARIATIONS

By MARSTON MORSE AND WILLIAM TRANSUE

1. **Introduction.** The point-wise convergence to s in the sense of Pringsheim of the double Fourier series of f (with values $f(s, t)$) is classically implied bv tests making use of the V-variation $V(Q, g)$ over the interval Q of functions g variously associated with f. (We write V-variation, F-variation, J-variation respectively for the Vitali, Fréchet and Jordan variation. The J-variation is defined for functions of a single variable only, and for such functions equals the V and F-variations). The functions g may be $f-s=\phi$ (Jordan-Hardy), $\overline{\phi/st}$ (Dini-Young), $\overline{st\phi}$ (Young), $\overline{\Delta_{ry}\phi/st}$ (Lebesgue-Gergen), etc. (If k is in L over all closed subintervals of Q, $\bar{k}$ shall denote an indefinite integral of k over Q). For example the Dini-Young condition takes the form

$$V\left[Q, \overline{\left(\frac{1}{st}\phi\right)}\right]=\int_Q\int\left|\frac{\phi(s, t)}{st}\right| dsdt<\infty \left[Q=(0, \pi]\times(0, \pi]\right]. \tag{1.1}$$

In the theory which we shall present in a subsequent paper any conditions on $V(Q, g)$ found in the classical theory will be replaced by the corresponding condition on the F-variation $P(Q, g)$ of g over Q. For example, the condition (1.1) will be replaced by the condition

$$P\left[Q, \overline{\left(\frac{1}{st}\phi\right)}\right]<\infty.$$

The F-variation has been studied by the authors as an essential adjunct of a theory of bilinear and quadratic functionals. On observing that the Dirichlet integral

$$\frac{1}{\pi^2}\int_0^\pi\int_0^\pi f(s, t)\frac{\sin (m+\frac{1}{2})s}{\sin (s/2)}\frac{\sin (n+\frac{1}{2})t}{\sin (t/2)}dsdt$$

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which gives the partial sum S_{mn} of the Fourier series of f is really a special evaluation of the bilinear functional

$$\Omega(\lambda, \mu) = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi f(s, t)\, \lambda(s)\, \mu(t) ds dt,$$

it became clear that theorems on such bilinear functionals obtained by the authors were relevant to the theory of Fourier series. As far as we know, the classical tests for the point-wise Pringsheim convergence of double Fourier series are all included in our tests on replacing $V(Q, g)$ (in its various forms) by $P(Q, g)$ That this inclusion is irreversible has been established.

The object of this paper is to develop a calculus for F-variations applicable to Fourier series theory and other branches of analysis. It should be recalled that the F-variation, unlike the V-variation, is not an additive function of an interval. Rather

$$P(Q_1 \cup Q_2, g) \leqslant P(Q_1, g) + P(Q_2, g)$$

whenever Q_1 and Q_2 intersect in a common edge. Moreover, no decomposition

$$P(s, t) - N(s, t) = g(s, t)$$

of a function g of finite F-variation into monotone components P and N is in general possible. These differences between F-and V-variations make new techniques necessary.

A generalization of the "second law of the mean" for functions of two variables is presented. Indefinite L-integrals over Q of functions ω defined over Q (open) lead us to a new type of indefinite integral $\bar{\omega}$ (u, v) termed an *FL-integral* and defined for $(u, v) \varepsilon \bar{Q}$. Our FL-integrals are Harnack integrals with special properties not shared by Harnack integrals in general. These FL-integrals $\bar{\omega}$ of ω may exist when the FL-integral of $|\omega|$ does not exist. Classical formulas for integration by parts are replaced by inequalities. Let $f_1 f_2 g$ with values $f_1(s) f_2(t)$ and $g(s, t)$ be given. Formulas for

$$P(Q, \overline{f_1 f_2 g}),\ P(Q, f_1 f_2 \bar{g}),\ P(Q, f_1 f_2 g)$$

are particularly important. In general one needs to know the effect of the removal or addition of the bar.

A proof of the effectiveness of the classical Lebesgue test involves a study of the integral

$$\int_x^\pi |\Delta_x g(t)|\, dt, \qquad (x>0)$$

where $g(t)$ has the form $\phi(t)/t$ This integral may be regarded as the J-variation of an indefinite integral of $\Delta_x g$ over the interval $[x, \pi]$. Generalizing we obtain a different but adequate theory of the F-variations of

$$\overline{\Delta_x\, \omega},\ \overline{\Delta_y\, \omega},\ \Delta_{xy}\, \omega,$$

where $\omega(s, t)$ has various forms such as $\phi(s, t)/st$.

We have previously [cf MT(5)] made an analysis of the boundary values of a functions g of finite F-variation $P(Q, g)$ over an open interval Q. To this we add here the theorem that $P(Q, g) \to 0$ as the maximum distance of the interval $J \subset Q$ from an edge of Q tends to zero.

Some but not all of our results have analogues for V-variations. These make the classical theory simpler and more transparent. The limitations of any theory based exclusively on Vitali variations become clear.

2. **Earlier theorems.** We are concerned with a Cartesian space R_2 of coordinates (s, t). An open interval in R_2 of points (s, t) of the form of a Cartesian product $(a, a') \times (b, b')$ of two open intervals will be denoted by $\begin{pmatrix} a', b' \\ a, b \end{pmatrix}$ The corresponding closed interval will be denoted by $\begin{bmatrix} a', b' \\ a, b \end{bmatrix}$. More generally we shall also *admit* intervals of the form $U \times V$, where U is any one of the intervals

$$(a, a')\ [a, a']\ (a, a']\ [a, a')$$

and V any one of the corresponding intervals in which b

and b' replace a and a'. If A is the point $s=a$ or $s=a'$ of $\bar{U}$ the product $A \times V$ is termed an *edge* of $Q = U \times V$. An edge $U \times B$ is similarly defined. An edge of Q may include one, both or neither of its end points.

Let g map a general interval $Q = U \times V$ into R_1, the axis of reals. If Q is closed the F-variation $P(Q, g)$ of g over Q is well defined. See Fréchet or MT (1). If Q is not closed one defines $P(Q, g)$ by the condition

$$P(Q, g) = \sup_J P(J, g) \tag{2.1}$$

taking the sup over all closed intervals $J \subset Q$. Then for $Q_1 \subset Q$

$$P(Q_1, g) \leqslant P(Q, g). \tag{2.2}$$

Moreover

$$P(Q_1 \cup Q_2, g) \leqslant P(Q_1, g) + P(Q_2, g) \tag{2.3}$$

when Q_1 and Q_2 intersect in a common edge. For functions g_1 and g_2 defined over Q,

$$P(Q, g_1+g_2) \leqslant P(Q, g_1) + P(Q, g_2). \tag{2.4}$$

If c is a constant and $P(Q, g) < \infty$

$$P(Q, cg) = |c| P(Q, g). \tag{2.5}$$

When $Q = \begin{bmatrix} a', b' \\ a\ , b \end{bmatrix}$ and $\begin{pmatrix} a', b' \\ a\ , b \end{pmatrix}$ we write

$$P(Q, g) = P\begin{bmatrix} a', b' \\ a\ , b \end{bmatrix}(g), \quad P(Q, g) = P\begin{pmatrix} a', b' \\ a\ , b \end{pmatrix}(g)$$

respectively. We now define a basic class of functions.

The class $\hat{F}(Q)$. Let K be a 1-dimensional straight line segment parallel to an edge of Q. The segment K need be neither closed nor open. If K is in Q, the function $g \mid K$ defined by g over K maps K into R_1 and the Jordan variation $T(K, g)$ of $g \mid K$ over K is well defined. We shall say that g is in $\hat{F}(Q)$ or satisfies $\hat{F}$ over Q if $P(Q; g) < \infty$, and if, for at least one section K of Q parallel to an arbitrary edge of Q, $T(K, g) < \infty$. If K and K' are two sections of Q parallel to the same edge of Q,

$$T(K, g) \leqslant T(K', g) + P(Q, g). \qquad (2.6)$$

See Fréchet or MT (5), relation (3.0). It follows from (2.6) that g is bounded when in $\hat{F}(Q)$.

Quadrant limits. Corresponding to each point (s_o, t_o) of R_2 the lines $s = s_o$, $t = t_o$ form the boundaries of four open regions (quadrants), $S(s_o, t_o)$ with vertex at (s_o, t_o). *If g satisfies $\hat{F}$ over Q and if the quadrant $S(s_o, t_o)$ intersects Q, then g has a limit as $(s, t) \to (s_o, t_o)$ in $S(s_o, t_o)$.* See MT (1). If (s_o, t_o) is in the interior of Q the four quadrant limits of g at (s_o, t_o) may be different. However, we have shown that when g satisfies $\hat{F}$ the points in Q at which g is discontinuous lie on a countable number of straight lines parallel to the edges of Q. See MT (1) Th. 6.3.

In these terms the natural generalization of the Jordan test is as follows. Suppose that f is defined over R_2 and has the period 2π in each variable. If f satisfies $\hat{F}$ over the interval $\begin{pmatrix} 2\pi, & 2\pi \\ 0, & 0 \end{pmatrix}$ the Fourier series of f converges at each point (s, t) in the sense of Pringsheim to the mean of the four quadrant limits of f at (s, t). See MT (8).

The extension g^c of g over $\bar{Q}$. Let g satisfy $\hat{F}$ over a general interval Q. For $(u, v) \in Q$ set $g^c(u, v) = g(u, v)$. If $Q \neq \bar{Q}$ let (u, v) be any point of $\bar{Q}$ not in Q. If (u, v) is on an edge E of $\bar{Q}$ let $g^c(u, v)$ be defined as the limit of $g(s, t)$ as (u, v) is approached on a section μ of Q orthogonal to E at (u, v), provided Q contains such a section. If (u, v) is a vertex of $\bar{Q}$ the section μ may not exist, and in this case $g(u, v)$ is taken as the quadrant limit of $g(s, t)$ at (u, v). *It then follows* (MT(5) Lemma 5.1) *for an arbitrary interval $J \subset \bar{Q}$ that*

$$P(J, g^c) = P(J \cap Q;\ g). \qquad (2.7)$$

According to this fundamental relation $P(J, g^e)$ is determined by the values of g over Q. In particular it follows that g^e satisfies $\hat{F}$ over $\bar{Q}$ if g satisfies $\hat{F}$ over Q.

From the definition of the extension g^e and the theorem on the existence of the quadrant limits one draws the following conclusion, useful for example in the study of our 2-dimensional de la Vallée Poussin test.

LEMMA 2.1. *If g satisfies $\hat{F}$ over a general interval Q and is continuous over Q, then its extension g^e exists, is continuous over $\bar{Q}$ and is in $\hat{F}(\bar{Q})$. Moreover*

$$P(Q, g) = P(\bar{Q}, g^e), \qquad \text{[cf. (2.7)]}.$$

Quadrant continuity. This is a generalization for the plane of left and right continuity for functions defined over a line. Let $S(s_o, t_o)$ be the first quadrant with vertex at (s_o, t_o). The function g defined over Q will be said to be *right continuous* at a point (s_o, t_o) of Q if $g(s_o, t_o)$ equals the limit of g as (s_o, t_o) is approached from $\bar{S}(s_o, t_o) \cap Q$. If (s_o, t_o) is the upper right vertex of Q this condition is vacuous; if (s_o, t_o) is an inner point of a right or upper edge E of Q, g is right continuous at (s_o, t_o) if $g \mid E$ is right continuous at (s_o, t_o). Continuity with respect to any other one of the four quadrant is similarly defined.

The reduced functions $g^{++}, g^{+-}, g^{-+}, g^{--}$. Suppose that g satisfies $\hat{F}$ over a closed interval Q. Let B be the boundary of Q. There exists a function g^{++} uniquely defined over Q by the following properties: $g^{++}(s, t) = g(s, t)$ at each point (s, t) of continuity of g and of $g \mid B$; and at the vertices of Q; g^{++} is right continuous at points of $Q-B$, and $g^{++} \mid B$ right continuous at inner points of edges of Q. Finally g^{-+}, g^{--}, g^{+-} are similarly defined using respectively the second, third, and fourth quadrants $S(s, t)$. Moreover

$$P(Q, g^{++}) = P(Q, g^{+-}) = P(Q, g^{-+}) = P(Q, g^{--}) \leqslant P(Q, g). \qquad (2.8)$$

For details see MT(1) Th. 6.4, where g^{++}, g^{-+}, g^{--}, g^{+-} are denoted by

$$g^{(1,2)},\ g^{(-1,2)},\ g^{(-1,-2)},\ g^{(1,-2)} \tag{2.9}$$

respectively. The condition found in MT(1) that k vanish on the left and lower edges of Q is not necessary to establish (2.8) provided g satisfy $\hat{F}$; if Q is the interval $\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$ the function k with values

$$k(s, t) = g(s, t) - g(a, t) - g(s, b) + g(a, b)\ [(s, t) \in Q] \tag{2.10}$$

vanishes on the left and lower edges of Q, satisfies $\hat{F}$ with g, and by MT(1) Th. 6.4 satisfies (2.8) (with k in place of g). But this replacement of g by k alters no variation appearing in (2.8), so that (2.8) holds as well for g as for k.

An evaluation of $P(Q, g)$, g reduced. Let $C[a, a']$ denote the Banach space of functions λ continuous on the interval $[a, a']$. For $\lambda \in C[a, a']$ and $\mu \in C[b, b']$ the iterated R-S-integral over the interval $Q = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$

$$\int_a^{a'} \lambda(s) d_s \int_b^{b'} \mu(t) d_t g(s, t) = B(\lambda, \mu, g) \tag{2.11}$$

is well defined provided g is in $\hat{F}(Q)$. Let $T_a^{a'}(f)$ denote the Jordan variation of f over the interval $[a, a']$. Let $\lambda \in C[a, a']$, with norm at most 1, be termed *subunit*. It follows from MT(2), equations (12.15) and (9.12), that *when g is reduced and in $\hat{F}(Q)$*

$$P(Q, g) = \sup_{\lambda, \mu} B(\lambda, \mu, g) \tag{2.12}$$

taking the sups for subunit $(\lambda, \mu) \in C[a, a'] \times C[b, b']$. In MT(2) the distribution functions g admitted are such that $g = g^{--}$, and g vanishes on the left and lower edges of Q. The latter condition does not affect the validity of (2.12) since each member of (2.12) is unchanged in value if g is replaced by k as defined in (2.10), while such a k

vanishes on the left and lower edges of Q. That the reduced functions g^{++}, g^{-+}, g^{+-}, g^{--} are all on an equal footing as far as equation (2.12) is concerned follows from the fact that an appropriate 1-1 linear maping of Q onto Q will carry any one of the reduced functions into one of the type g^{--} without altering the values of the members of (2.12).

Equation (2.12) *holds in particular when* $P(Q, g) < \infty$ *and* g *is continuous.*

Let $C^*[a, a']$ denote the subset of elements $\lambda \varepsilon C[a, a']$ such that $\lambda(a) = \lambda(a')$. We shall need the following special evaluation of $P(Q, g)$, where g is continuous.

LEMMA 2.2. *If* g *is continuous over* $Q = \begin{bmatrix} a', & b' \\ a\ , & b \end{bmatrix}$ *and* $g \,\varepsilon\, \hat{F}(Q)$ *then*

$$P(Q, g) = \sup_{\lambda, \mu} \int_a^{a'} \lambda(s) ds \int_b^{b'} \mu(t) d_t g(s, t) \tag{2.13}$$

taking the sup *over all subunit* $(\lambda, \mu) \,\varepsilon\, C^*[a, a'] \times C^*[b, b']$.

PROOF. If h (with values $h(s)$) has a finite Jordan variation over $[a, a']$ and is continuous, the sup of the R-S-integral

$$\int_a^{a'} \lambda(s) dh(s) \tag{2.14}$$

is the same if taken over all subunit $\lambda \,\varepsilon\, C[a, a']$ as over all subunit $\lambda \,\varepsilon\, C^*[a, a']$. To apply this principle to (2.13) we need to know that

$$\int_b^{b'} \mu(t) d_t g(s, t) = h_g(s) \qquad \{ s \,\varepsilon [a, a'] \} \tag{2.15}$$

is continuous over $[a, a']$, under the hypotheses of the lemma. To this end let $s' < s''$ be values in $[a, a']$. By a theorem of F. Riesz

$$\begin{aligned} |h_g(s'') - h_g(s')| &\leqslant T_b^{b'}[g(s'', \cdot) - g(s', \cdot)] \\ &\leqslant P\begin{bmatrix} s'', & b' \\ s'\ , & b \end{bmatrix}(g) \end{aligned} \tag{2.16}$$

in accordance with the definition of the F-variation.

(The function defined by $g(s, t)$ when t is constant and s variable is denoted by $g(\cdot, t)$.) It follows from Cor. 3.5 of the next section that the right member of (2.15) is less than a prescribed $e > 0$, if $|s''-s'| < \delta_e$ and δ_e is sufficiently small.

Hence the integral (2.15) varies continuously with $s \epsilon [a, a']$, and the theorem stated for (2.14) implies that the sup in (2.13) is the same for subunit $(\lambda, \mu) \epsilon C^*[a, a'] \times C[b, b']$ as for subunit $(\lambda, \mu) \epsilon C[a, a'] \times C[b, b']$. But as Fréchet has shown, when g is in $\hat{F}(Q)$

$$\int_a^{a'} \lambda(s) d_s \int_b^{b'} \mu(t) d_t g(s, t) = \int_b^{b'} \mu(t) d_t \int_a^{a'} \lambda(s) d_s g(s, t). \quad (2.17)$$

We can accordingly reason with the right member of (2.17) as with the left, and using the theorems on the integral (2.14) infer that the sup of the right member of (2.17) is the same for subunit $(\lambda, \mu) \epsilon C^*[a, a'] \times C[b, b']$ as for subunit $(\lambda, \mu) \epsilon C^*[a, a'] \times C^*[b, b']$. The lemma follows.

...easurability of $g \epsilon \hat{F}(Q)$. We need consider only the case in which Q is open. As stated previously there exists a uniq.e function g^{--} which is left continuous over Q and such that $g^{--}(s, t) = g(s, t)$ at each point of continuity of g. Moreover the points of discontinuity of g all lie on a countable number of straight lines parallel to the edges of Q. Hence $g^{--}(s, t) = g(s, t)$ except at most on a set of measure zero. Cf. MT(1) Th. 6.3. In MT (6) §2 we have shown that a function such as g^{--} is measurable. Hence g is measurable, when in $\hat{F}(Q)$. A function $g \epsilon \hat{F}(Q)$ is bounded and measurable and hence integrable.

3. **Limiting values of $P(J, g)$ for variable interval J.** We recall the notation used in defining $P(Q, g)$ over a

closed interval $\begin{bmatrix} a', b' \\ a, b \end{bmatrix}$. For that purpose let π be a partition of the intervals $[a\ a']$ and $[b, b']$ of the form

$$a = s_0 < s_1 < \ldots < s_{r(\pi)} = a' \qquad (3.1)'$$

$$b = t_0 < t_1 < \ldots < t_{n(\pi)} = b'. \qquad (3.1)''$$

If g maps Q into R_1 one sets

$$\Delta_{rn}(g) = g(s_r, t_n) - g(s_{r-1}, t_n) - g(s_r, t_{n-1}) + g(s_{r-1}, t_{n-1}).$$

Constants e_r with $|e_r| \leqslant 1$ are *associated* with the respective subintervals of $[a, a']$. We have

$$P[Q, g] = \sup_{\pi e} \sum_n |e_r \Delta_{rn}(g)| \begin{Bmatrix} r = 1, \ldots, r(\pi) \\ n = 1, \ldots, n(\pi) \end{Bmatrix}, \qquad (3.2)$$

where a summation as to r is understood as in tensor algebra, and where the sup is over all partitions π of Q and associated sets

$$\mathrm{e} = (e_1, \ldots, e_{r(\pi)}) \qquad (|e_r| \leqslant 1).$$

Equation (3.2) is not the original definition of $P(Q, g)$ but is easily seen to be equivalent to that definition. Cf. MT(5) Lemma 2.1.

THEOREM 3.1. *Let g map a closed interval Q into R_1 in a manner such that $P(Q, g) < \infty$. Let E be an arbitrary edge of Q and J a subinterval of Q not intersecting E. Then $P(J, g) \to 0$ as the maximum distance of the vertices of J from E tends to zero.*

We proceed by assuming the theorem to be false, taking Q as the interval $\begin{bmatrix} a', b' \\ a, b \end{bmatrix}$ and E as the edge of Q on which $s = a$. We infer the existence of a constant δ with the following property. There exist intervals $J_s \subset Q$ of the form

$$J_s = (a, a+s) \times [b, b']$$

such that

$$P(J_s, g) > \delta \qquad (3.3)$$

for arbitrarily small $s > 0$.

By virtue of the definition of an F-variation over a non-closed interval such as J_s, there exist three (or arbi-

trarily many, if we please) closed non-intersecting intervals $J^{(i)}(i = 1, 2, 3)$ of the form

$$J^{(i)} = \begin{bmatrix} a_i', & b' \\ a_i, & b \end{bmatrix} \quad [a < a_1 < a'_1 < a_2 < a_2' < a_3 < a_3' < s_0]$$

for which $s_0 > 0$ is arbitrarily small and

$$P(J^{(i)}, g) > \delta. \tag{3.4}$$

There accordingly exists a partition $\pi^{(i)}$ of $J^{(i)}$, an associated set of constants,

$$\mathrm{e}^{(i)} = (e_1^{(i)}, \ldots, e^{(i)}_{r(\pi^{(i)})}) \quad (i = 1, 2, 3)$$

and double differences $\Delta^{(i)}_{rn}(g)$ such that

$$\sum_n |e_r^{(i)} \; \Delta^{(i)}_{rn}(g)| > \delta, \quad \begin{matrix} (r = 1, \ldots . r(\pi^{(i)}) \\ (n = 1, \ldots, n(\pi^{(i)}) \end{matrix} \tag{3.5}$$

where r and n are, summed, but not i. Without loss of generality we can suppose that the partitions $\pi^{(i)}$ of $[b, b']$ are independent of i. For one can replace the partition $\pi^{(i)}$ of $[b, b']$ by a partition which includes all the vertices of $\pi^{(1)}$, $\pi^{(2)}$, $\pi^{(3)}$, assigning to any new subinterval h of an original interval k of $\pi^{(i)}$ a constant $e_r^{(i)}$ equal to the constant originally assigned to k. The left members of (3.5) $(i = 1, 2, 3)$ will thereby be undiminished. With this understood set the common $n(\pi^{(i)}) = m$, and set

$$e_r^{(i)} \Delta^{(i)}_{rn}(g) = A^{(i)}_n (n = 1, \ldots, m; \; i \text{ not summed}). \tag{3.6}$$

We shall define a partition Π of the interval $\begin{bmatrix} a_3', & b' \\ a_1, & b \end{bmatrix}$

and shall associate constants ϵ_j, $(j = 1, \ldots, p)$ with the resulting subintervals of $[a_1, a_3']$. Under Π the three subintervals $[a_i, a_i']$ shall receive the respective partitions $\pi^{(i)}$, while the two remaining intervals, $[a_1', a_2]$ and $[a_2', a_3]$, shall not be further subdivided. Let $c_i (i = 1, 2, 3)$ be constants with $|c_i| \leqslant 1$. We associate the constants $c_i e_r^{(i)}$ (i not summed) with the $r(\pi^{(i)})$ subintervals of $[a_i, a_i']$, and constants 0 with the intervals $[a_1', a_2]$ and $[a_2', a_3]$. For this partition Π, associated constants e_r, and double differences $\Delta_{rn}(g)$

$$P\begin{bmatrix} a'_3, b' \\ a_1, b \end{bmatrix}(g) \geqslant \sum_n |e_r \Delta_{rn}(g)| = \sum_n |c_1 A_n^{(1)} + c_2 A_n^{(2)} + c_3 A_n^{(3)}|,$$

when $n = 1, \ldots, m$. By virtue of (3.5), for $i = 1, 2, 3$,

$$\sum_n |A_n^{(i)}| > \delta \quad (n = 1, \ldots, m).$$

We turn to Lemma 3.1 (which follows) and infer that for suitable choices of c_1, c_2, c_3

$$P\begin{bmatrix} a'_3, b' \\ a_1, b \end{bmatrix}(g) \geqslant \frac{13}{12}\delta.$$

There must then exist intervals J_s for which s is arbitrarily small and $P(J_s, g) \geqslant 13\delta/12$. We can accordingly replace δ in (3.4) by $13\delta/12$. If N is any positive integer, the existence of intervals J_s with s arbitrarily small and

$$P(J_s, g) \geqslant (13/12)^N \delta$$

is implied. Since this is contrary to the hypothesis that $P(Q, g) < \infty$ the theorem follows.

COROLLARY 3.1. *The preceding theorem holds for a general interval Q.*

To establish the corollary one has merely to note that the hypothesis $P(Q, g) < \infty$ implies that $P(\bar{Q}, g^e) < \infty$, where the extension g^e of g is defined over $\bar{Q}$ in §2, and that by virtue of (2.7), and Th. 3.1 applied to $\bar{Q}$,

$$P(J \cap Q, g) = P(J, g^e) \to 0,$$

under the conditions on J of Th. 3.1.

The preceding theorem and corollary admit a generalization to μ-intervals Q in μ-space R_μ, $\mu > 2$, keeping E in Th. 3.1 a 1-dimensional edge of $\bar{Q}$. In the proof of the theorem for $\mu > 2$ the above linear sums $e_r^{(i)} \Delta_{rn}^{(i)}$ must be replaced by multilinear sums of dimension $\mu - 1$, with i again 1, 2, 3. Lemma 3.1 is to be applied as in the case $\mu = 2$.

COROLLARY 3.2. *Suppose that $P(Q, g) < \infty$ for a general interval Q. Let E be an edge of $\bar{Q}$ and (s, t) and (s', t') points of Q not in E, with $s' > s, t' > t$. Given e there exists a $\delta > 0$ such that*

$$g(s', t')-g(s', t)-g(s, t')+g(s, t) < e$$

provided only that the distance of (s', t') and (s, t) from E is less than δ.

COROLLARY 3 3. *Suppose that g satisfies $\hat{F}$ over a general interval Q. Let E be an edge of $\bar{Q}$ and J_y a subinterval of Q with E in $\bar{J}_y$ but not in J_y, and with width y in the direction orthogonal to E. If K_y is any section of J_y orthogonal to E*

$$T(K_y, g) \to 0 \quad [\text{as } y \to 0]$$

uniformly for all sections K_y of J_y.

Let K'_y be any section of J_y parallel to K_y. Then by (2.6)

$$T(K_y, g) \leqslant T(K'_y, g)+P(J_y, g). \tag{3.7}$$

If the projection of K'_y in E is fixed, both variations on the right of (3.7) tend to zero with y and Cor. 3.3 follows.

In Th. 3.1 and Cor. 3.1 it is assumed that the subinterval J of Q does not intersect the edge E of Q. The following corollary gives a condition under which J may contain E and the earlier conclusions still hold. This corollary involves the condition that g take on continuous *normal boundary values* on E, that is, that at any point (a, b) on E, $g(a, b)$ equals the limit of $g(u, v)$ as (a, b) is approached on a normal to E at (a, b). The function $g \mid E$ need not be continuous.

COROLLARY 3 4. *Let Q be a general interval including an edge E and suppose that $P(Q, g) < \infty$ and that g takes on continuous normal boundary values on E. If J is an arbitrary subinterval of Q containing E, $P(J, g) \to 0$ as the area of J tends to zero.*

By an obvious extension of Lemma 5 1 of MT (5) $P(J, g) = P(J-E, g)$ and Cor. 3.4 follows from Cor. 3.1.

If F is continuous over a closed interval Q then any closed subinterval Q_1 of Q with an edge E satisfies the conditions on Q and E in Cor. 3.4. An obvious use of the Heine-Borel theorem gives the following.

COROLLARY 3.5. *Suppose that g is continuous over a closed interval Q with $P(Q, g) < \infty$. If J is an arbitrary subinterval of Q, $P(J, g) \to 0$ as the area of J tends to zero, uniformly for all subintervals J of Q.*

COROLLARY 3.6. *Suppose that g is continuous over a general interval Q and satisfies $\hat{F}$. Let Q_n be any sequence of subintervals of Q whose areas tend to the area of Q as $n \to \infty$. Then $P(Q_n, g) \to P(Q, g)$* [*as* $n \to \infty$].

The interval Q is the union of Q_n and at most four subrectangles J_n the distances of whose vertices from an edge of Q tend to zero as $n \to \infty$. For simplicity suppose that there is just one such rectangle J_n. Then one can suppose that J_n and Q_n have an edge in common so that

$$P(Q, g) \leqslant P(Q_n, g) + P(J_n, g).$$

Since

$$P(Q, g) - P(J_n, g) \leqslant P(Q_n, g) \leqslant P(Q, g)$$

and $P(J_n, g) \to 0$ as $n \to \infty$ by virtue of Cor. 3.4 or Cor. 3.1, Cor. 3.6 follows as stated.

Note. Cor. 3.6 *should be compared with* Th. 10.1.

The lemma used in proving Th. 3.1 follows.

LEMMA 3.1. *Let δ be a positive constant and $A_1^{(i)} \ldots A_m^{(i)}$ $(i = 1, 2, 3)$ three sets of constants such that*

$$\sum_n |A_n^{(i)}| > \delta \quad (n = 1, \ldots, m). \tag{3.8}$$

Then

$$\sum_n |c_1 A_n^{(1)} + c_2 A_n^{(2)} + c_3 A_n^{(3)}| > \delta + \delta/12 \tag{3.9}$$

for some choice of c_1, c_2, c_3 with $|c_i| \leqslant 1$.

We may assume that

$$\sum_n |A_n^{(1)} + A_n^{(2)}| \leqslant \delta + \delta/12, \quad \sum_n |A_n^{(1)} - A_n^{(2)}| \leqslant \delta + \delta/12, \tag{3.10}$$

for otherwise (3.9) would hold for obvious choices of (c_1, c_2, c_3).

Let Σ' denote any sum of elements a_n for those values of n for which sign $A_n^{(1)} =$ sign $A_n^{(2)}$ and let Σ'' denote a

sum of the remaining elements a_n. (We take sign $0 = 1$). Then from (3.8) and (3.10), we have respectively

$$\Sigma' |A_n^{(1)}+A_n^{(2)}| + \Sigma'' |A_n^{(1)}-A_n^{(2)}| = \Sigma |A_n^{(1)}| + \Sigma |A_n^{(2)}| > 2\delta \quad (3.11)$$

$$\Sigma |A_n^{(1)}+A_n^{(2)}| + \Sigma |A_n^{(1)}-A_n^{(2)}| \leqslant 2\delta + \delta/6. \quad (3.12)$$

Subtracting corresponding members of (3.11) and (3.12), we have

$$\Sigma'' |A_n^{(1)}+A_n^{(2)}| + \Sigma' |A_n^{(1)}-A_n^{(2)}| < \delta/6 \quad (3.13)$$

noting that the left member of (3.11) is at most the left member of (3.12), that is, at most $2\delta+\delta/6$. We continue by distinguishing the cases.

Case I. $\Sigma' |A_n^{(3)}| > \delta/2$;

Case II. $\Sigma'' |A_n^{(3)}| > \delta/2$,

covering all possibilities in accordance with (3.8).

Case I. For $e = \pm 1$, by virtue of (3.13)

$$\Sigma' |A_n^{(1)}-A_n^{(2)}+eA_n^{(3)}| \geqslant \Sigma' |A_n^{(3)}| - \Sigma' |A_n^{(1)}-A_n^{(2)}|$$
$$> \delta/2 - \delta/6 = \delta/3. \quad (3.14)$$

Using (3.11) and (3.10), we have

$$\Sigma'' |A_n^{(1)}-A_n^{(2)}| > 2\delta - \Sigma' |A_n^{(1)}+A_n^{(2)}|$$
$$> 2\delta - (\delta+\delta/12) = \delta - \delta/12 \quad (3.15)$$

$$\Sigma'' |A_n^{(1)}-A_n^{(2)}+A_n^{(3)}| + \Sigma'' |A_n^{(1)}-A_n^{(2)}-A_n^{(3)}|$$
$$\geqslant \Sigma'' 2 |A_n^{(1)}-A_n^{(2)}| > 2\delta - \delta/6 \quad \text{[from (3.15)]}.$$

Hence for at least one choice of $e_0 = \pm 1$,

$$\Sigma'' |A_n^{(1)}-A_n^{(2)}+e_0A_n^{(3)}| > \delta - \delta/12. \quad (3.16)$$

It follows from (3.14) and (3.16) that

$$\Sigma |A_n^{(1)}-A_n^{(2)}+e_0A_n^{(3)}| > \delta/3 + (\delta-\delta/12) = \delta+\delta/4.$$

This establishes (3.9) in *Case I*, assuming (3.10).

The proof of (3.9) is similar in *Case II*, interchanging $A_n^{(2)}$ and $-A_n^{(2)}$, as well as Σ' and Σ'', in the above proof under *Case I*.

Other representations of $P(Q, g)$. We shall close this section with a modification of the representation (3.2) of

$P(Q, g)$. We refer to the partition π of Q defined by (3.1), and set

$$g(s_r, t) - g(s_{r-1}, t) = \Delta_r^{(1)} g(\cdot, t). \tag{3.17}$$

The function mapping $[b, b']$ into R_1 with value at t given by (3.17) will be denoted by $\Delta_r^{(1)} g$, the superscript (1) making clear which argument is being differenced. Similarly $\Delta_n^{(2)} g$ is a function mapping $[a, a']$ into R_1 with values

$$g(s, t_n) - g(s, t_{n-1}) = \Delta_n^{(2)} g(s, \cdot). \tag{3.18}$$

We shall also apply the operators $\Delta_r^{(1)}$ to any function which is defined over $[a, a']$ and apply $\Delta_n^{(2)}$ similarly to any function defined over $[b, b']$. In particular it is clear that

$$\Delta_{rn} g = \Delta_r^{(1)} \Delta_n^{(2)} g = \Delta_n^{(2)} \Delta_r^{(1)} g.$$

With this notation (3.2) can be given the alternative form

$$P(Q, g) = \sup_{\pi, e} \sum_n | \Delta_n^{(2)} [e_r \Delta_r^{(1)} g] | \quad \begin{pmatrix} n = 1, \ldots, n(\pi) \\ r = 1, \ldots, r(\pi) \end{pmatrix}$$

$$= \sup_{\pi, e} T_b^{b'} (e_r \Delta_r^{(1)} g). \tag{3.19'}$$

As stated, (3.19)′ is limited to closed intervals Q. If, however, one recalls the definition of the F-variation over non-closed intervals this limitation can be removed.

One has merely to understand that a partition π of a non-closed interval is a partition of some closed subinterval I_π of Q and that the set e *is associated with the subintervals of I_π.* Suppose that Q is a general interval in the form of a Cartesian product $U \times V$. Then

$$P(Q, g) = \sup_{\pi, e} T(V, e_r \Delta_r^{(1)} g) \tag{3.19''}$$

including (3.19)′ as a special case. It is understood that I_π ranges over all closed subintervals of U. Formulas (3.19) do not require that $P(Q, g)$ be finite.

4. **Products** $h(s)k(t)\phi(s, t)$. We shall be concerned with functions h, k, f, ϕ, g with values $h(s)$, $k(t)$, $f(t)$, $\phi(s, t)$, $g(s, t)$ over intervals to be specified. A preliminary lemma is 1-dimensional.

LEMMA 4.1. *Suppose that* $T_o^{b'}(f) < \infty$ *and* $f(0) = 0$. *With* $0 \leqslant b < b'$ *let* k *map* $[b, b']$ *continuously into* R_1, *with* $T_b^{b'}(k) < \infty$. *Then*

$$T_b^{b'}(kf) \leqslant |k(b')|\, T_b^{b'}(f) + 2\int_b^{b'} T_o^t(f)\, |dk(t)|. \quad (4.1)$$

COROLLARY 4.1. *If* $T_o^t(f) \leqslant At$ *over* $[0, b']$ *for some constant* A *and if* $f(0) = 0$ *then for fixed* b' *and variable* b, *with* $0 < b < b'$

$$T_b^{b'}(ft^{-2}) < AK/b, \quad (4.7)$$

where K *is independent of* A.

We obtain (4.7) from (4.1) on setting $k(t) = t^{-2}$ in (4.1).

LEMMA 4.2. *Let* g *be defined over* $Q = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$ *and* k, *with values* $k(t)$ *be defined over* $[b, b']$. *If* $g(s, b) \equiv 0$ *then*

$$P(Q, kg) \leqslant [\sup_t |k(t)| + T_t^{b'}(k)]\, P(Q, g), \quad (4.8)$$

admitting the possibility that $P(Q, g) = \infty$.

If f and k are defined over $[b, b']$ it is easy to show that

$$T_b^{b'}(fk) \leqslant \sup_t |k(t)|\, T_b^{b'}(f) + \sup_t |f(t)|\, T_b^{b'}(k). \quad (4.9)$$

To establish (4.8) we refer to a partition π of $[b, b']$ and recall that

$$P(Q, kg) = \sup_{\pi, e} T_b^{b'}[e_r \Delta_r^{(1)}(kg)] \quad [\text{as in } (3.19)']. \quad (4.10)$$

Let f be the function mapping $[b, b']$ into R_1 with values

$$f(t) = e_r \Delta_r^{(1)} g(\cdot, t) \quad (4.11)$$

for fixed π and e. Then (4.9) holds. Since $g(s, b) = 0$, $f(b) = 0$ so that

$$f(t) \leqslant T_b^{b'}(f) \leqslant P(Q, g) \quad [\text{using } (3.19)']. \quad (4.12)$$

Hence (4.9), taken for f as defined in (4.11), implies (4.8).

Lemma 4.2 holds with V replacing P. One starts with (4.9) as above and uses the fact that

$$V(Q, kg) = \sup_\pi \sum_r T_b^{b'}[\Delta_r^{(1)}(kg)]$$

and proceeds essentially as above. This relation will not be used.

COROLLARY 4.2. *Let ϕ satisfy $\hat{F}$ over Q and k be of bounded J-variation over $[b, b']$. Then $k\phi$ satisfies $\hat{F}$ over Q.*

The corollary follows from the lemma on noting that $\phi(s, t)-\phi(s, b) = g(s, t)$ vanishes when $t = b$. Hence $P(Q, kg) < \infty$ as a consequence of (4.8). With $\phi(s, b) = h(s)$ $T_a^{a'}(h) < \infty$, since ϕ satisfies $\hat{F}$ over Q. Moreover

$$P(Q, hk) = T_a^{a'}(h)\,T_b^{b'}(k) < \infty.$$

Since $\phi = g+h$

$$P(Q, k\phi) \leqslant P(Q, hk)+P(Q, kg) < \infty.$$

If K is any section of Q parallel to an edge of Q, $T(K, \phi \mid K) < \infty$ since ϕ satisfies $\hat{F}$, and it follows that $T(K, k\phi \mid K) < \infty$.

LEMMA 4.3. *Suppose $P(I, g) < \infty$ for $I = \begin{bmatrix} a', & b' \\ a, & 0 \end{bmatrix}$, with $g(s, 0) = 0$. For $Q = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix} \subset I$ let k map $[b, b']$ continuously into R_1 with $T_b^{b'}(k) < \infty$. Then*

$$P(Q, kg) \leqslant |\,k(b')\,|\,P\begin{bmatrix} a', & b' \\ a, & 0 \end{bmatrix}(g)+2\int_b^{b'} P\begin{bmatrix} a', & t \\ a, & 0 \end{bmatrix}(g)\,|\,dk(t)\,|. \tag{4.14}$$

We shall make use of (3.19)′ setting

$$e_r \Delta_r^{(1)} g(\cdot, t) = f(t) = f(t; \pi, e) \tag{4.15}$$

for fixed partition π of $[a, a']$ and e. Note that $f(0) = 0$ since $g(s, 0) \equiv 0$. By Lemma 4.1 we have

$$T_b^{b'}(fk) \leqslant |\,k(b')\,|\,T_b^{b'}(f)+2\int_b^{b'} T_o^t(f)\,|\,dk(t)\,|. \tag{4.16}$$

Referring to (3.19)′

$$T_b^t(f) \leqslant P\begin{bmatrix} a', & t \\ a, & b \end{bmatrix}(g) \leqslant P\begin{bmatrix} a', & t \\ a, & 0 \end{bmatrix}(g) \tag{4.17}$$

so that (4.16) gives

$$T_b^{b'}(fk) \leqslant |k(b')|P\begin{bmatrix} a', & b' \\ a, & 0 \end{bmatrix}(g)+2\int_b^{b'} P\begin{bmatrix} a', & t \\ a, & 0 \end{bmatrix}(g)\,|dk(t)|. \tag{4.18}$$

Using (3.19)′ and taking the sup of the left number of (4.18) over all admissible partitions π and associated sets e, (4.14) follows.

The preceding lemma leads to the following theorem.

THEOREM 4.1. *Suppose that* $P(I,\phi) < \infty$ *for* $I = \begin{bmatrix} a', & b' \\ 0, & 0 \end{bmatrix}$ *and that* ϕ *vanishes on the left and lower edges of* I. *With* $Q = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix} \subset I$ *suppose that* h *and* k *are continuous and of bounded* J*-variation over* $[a, a']$ *and* $[b, b']$ *respectively. Then*

$$\begin{aligned} P(Q, hk\phi) \leqslant{} & |h(a')|\,|k(b')|P(I, \phi) \\ & +2|h(a')|\int_b^{b'} P\begin{bmatrix} ', & t \\ 0, & 0 \end{bmatrix}(\phi)\,|dk(t)| \\ & +2|k(b')|\int_a^{a'} P\begin{bmatrix} s, & b' \\ 0, & 0 \end{bmatrix}(\phi)\,|dh(s)| \\ & +4\int_a^{a'} |dh(s)|\int_b^{b'} P\begin{bmatrix} s, & t \\ 0, & 0 \end{bmatrix}(\phi)\,|dk(t)|. \end{aligned} \tag{4.19}$$

On interchanging the roles of s and t in Lemma 4.3 we have

$$P\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(h\phi) \leqslant |h(a')|P\begin{bmatrix} a', & b' \\ 0, & b \end{bmatrix}(\phi)+2\int_a^{a'} P\begin{bmatrix} s, & b' \\ 0, & b \end{bmatrix}(\phi)|dh(s)|. \tag{4.20}$$

It is clear (Cor. 4.2) that $h\phi$ satisfies $\hat{F}$ over I. We can accordingly apply Lemma 4.3 again with $g = h\phi$. Thus

$$P(Q, hk\phi) \leqslant |k(b')|P\begin{bmatrix} a', & b' \\ a, & 0 \end{bmatrix}(h\phi)+2\int_b^{b'} P\begin{bmatrix} a', & t \\ a, & 0 \end{bmatrix}(h\phi)\,|dk(t)|. \tag{4.21}$$

Relation (4.20) with $[b, b']$ replaced by $[0, t]$ gives

$$P\begin{bmatrix} a', & t \\ a, & 0 \end{bmatrix}(h\phi) \leqslant |h(a')|P\begin{bmatrix} a' & t \\ 0, & 0 \end{bmatrix}(\phi)+2\int_a^{a'} P\begin{bmatrix} s, & t \\ 0 & 0 \end{bmatrix}(\phi)\,|dh(s)|. \tag{4.22}$$

On substituting the right member of (4.22) appropriately in (4.21), (4.19) is obtained.

COROLLARY 4.3. *If ϕ in Th. 4.1 is such that*

$$P\begin{bmatrix} s, & t \\ 0, & 0 \end{bmatrix}(\phi) \leqslant Ast, \quad (A \text{ a constant}) \tag{4.23}$$

then for fixed a', b' and variable $a > 0$, $b > 0$,

$$P\begin{bmatrix} a', & b' \\ a\ , & b \end{bmatrix}\left(\frac{\phi}{s^2t^2}\right) \leqslant \frac{AK}{ab}, \tag{4.24}$$

where K is independent of A.

Relation (4.24) is a consequence of (4.19) on setting $h(s) = s^{-2}$ and $k(t) = t^{-2}$.

Corollary 4.3 will be used in the proof that our generalization of Young's condition for the convergence of a double Fourier series implies our generalization of the Lebesgue-Pollard conditions and in other connections.

Lemma 4.3 and Th. 4.1 are readily established with V replacing P, relying on the general relation

$$V(Q, g) = \sup_{\pi} \sum_{r} T_b^{b'}(\Delta_r^{(1)} g)$$

and using Lemma 4.1.

5. **A generalized second law of the mean.** Let f be in $L[b, b']$, that is, integrable over $[b, b']$ and set

$$M_b^{b'}(f) = \max_t \left| \int_t^{b'} f(t)\,dt \right| \quad \{t \epsilon [b, b']\}. \tag{5.1}$$

If γ maps the open interval (b, b') into R_1 with $T_{b\,+}^{b'-}(\gamma) < \infty$ and $\gamma(b+) = 0$ then

$$\left| \int_b^{b'} f(t)\gamma(t)\,dt \right| \leqslant M_b^{b'}(f)\, T_{b\,+}^{b'-}(\gamma), \tag{5.2$'$}$$

where $T_{b\,+}^{b'-}$ is the total variation of γ over (b, b'). Relation (5.2)$'$ implies that

$$\left| \int_b^{b'} f(t)\gamma(t)\,dt \right| = \left| \int_\xi^{b'} f(t)\,dt \right| T_{b\,+}^{b'-}(\gamma) \tag{5.2$''$}$$

for some $\xi \epsilon [b, b']$. Relations (5.2) are a consequence of the second law of the mean and in most applications [It is obvious that the roles of b and b' can be interchanged

replacing the condition $\gamma(b+)=0$ by $\gamma(b'-)=0$).] can replace the second law of the mean. We have obtained a generalization of (5.2)′ for multiple integrals. See MT(5) §9. One can drop the condition $\gamma(b+)=0$ under which (5.2)′ has been stated as follows.

LEMMA 5.1. *If f is in $L[b, b']$ and $T_b^{b'+}{}_-(\gamma)<\infty$, then*

$$\Big|\int_b^{b'} f(t)\gamma(t)dt\,\Big| \leqslant M_b^{b'}(f)\left\{T_{b\,+}^{b'-}(\gamma)+|\gamma(b+)|\right\}. \tag{5.3}$$

Lemma 5.1 and Cor. 4.1 combine to give the following corollary.

COROLLARY 5.1. *If f is in $L[0, b']$ and $T_0^t(\gamma)\leqslant At$ for $t\,\varepsilon\,[0, b']$ for some constant A, and if $\gamma(0)=0$, then for fixed b' and variable b, with $0<b<b'$*

$$J=\Big|\int_b^{b'} f(t)\frac{\gamma(t)}{t^2}\,dt\,\Big| \leqslant M_b^{b'}(f)\frac{AK}{b}, \tag{5.4}$$

where K is independent of $M_b^{b'}(f)$ and A.

In accordance with (5.3) for $0<b<b'$

$$J\leqslant M_b^{b'}(f)\ \left\{T_b^{b'}\Big(\frac{\gamma}{t^2}\Big)+\Big|\frac{\gamma(b+)}{b^2}\Big|\right\}$$

and (5.4) follows with the aid of Cor. 4.1 and the relat on $|\gamma(t)|\leqslant At$.

For proof of the following generalization of (5.2), see MT(5) Th. 9.4.

LEMMA 5.2. *Given $f_1\,\varepsilon\,L[a\ a'], f_2\,\varepsilon\,L[b, b']$, if g satisfies $\hat{F}$ over $Q=\begin{pmatrix} a', & b' \\ a, & b\end{pmatrix}$ and if $g(a+, t)=0$ for $t\,\varepsilon\,(b, b')$ and $g(s, b+)=0$ for $s\,\varepsilon\,(a, a')$ then*

$$\Big|\int_a^{a'}\int_b^{b'} f_1(s)f_2(t)g(s, t)dsdt\,\Big| \leqslant M_a^{a'}(f)\,M_b^{b'}(f_2)P(Q, g).$$

The following theorem is the 2-dimensional analogue of Lemma 5.1 and is based on Lemma 5.2. Denote the functions with values $g(a+, t)$ and $g(s, b+)$ by $g(a+, \cdot)$ and $g(\cdot, b+)$ respectively. The quadrant limit of $g(s, t)$ at (a, b) will be denoted by $g(a+, b+)$.

THEOREM 5.1. *Given $f_1 \varepsilon L[a, a']$, $f_2 \varepsilon L[b, b']$, if g satisfies $\hat{F}$ over $Q = \begin{pmatrix} a', b' \\ a, b \end{pmatrix}$, then*

$$\left| \int_a^{a'} \int_b^{b'} f_1(s) f_2(t) g(s, t)\, ds\, dt \right| = \mathcal{J} \tag{5.6}$$

$$\leqslant M_a^{a'}(f_1) M_b^{b'}(f_2) \{ P(Q, g) + T_{a\,+}^{a'\,-}[g(\cdot, b+)] + T_{b\,+}^{b'\,-}[g(a+, \cdot)] + |g(a+, b+)| \}.$$

To establish this theorem set

$$\begin{aligned} g_0 &= g(a+, b+) \\ g_1(s) &= g(s, b+) - g_0 \\ g_2(t) &= g(a+, t) - g_0 \\ g_3(s, t) &= g(s, t) - g_1(s) - g_2(t) - g_0 \end{aligned} \qquad [(s, t) \,\varepsilon\, Q]$$

and note that

$$g(s, t) = g_0 + g_1(s) + g_2(t) + g_3(s, t). \tag{5.7}$$

Observe that $g_3(a+, t) = 0$ for $t \,\varepsilon (b, b')$ and $g_3(s, b+) = 0$ for $s \,\varepsilon\, (a, a')$. In verifying this, use has been made of the property of the quadrant limit $g(a+, b+)$ that

$$\lim_{t \to b+} g(a+, t) = \lim_{s \to a+} g(s, b+) = g(a+, b+).$$

Since g_3 satisfies the conditions on g in Lemma 5.2

$$\left| \int_a^{a'} \int_b^{b'} f_1(s) f_2(t) g_3(s, t)\, ds\, dt \right| \leqslant M_a^{a'}(f_1) M_b^{b'}(f_2) P(Q, g_3). \tag{5.8}$$

From (5.2) it follows that

$$\left| \int_a^{a'} f_1(s) g_1(s)\, ds \right| \leqslant M_a^{a'}(f_1) T_{a\,+}^{a'\,-}(g_1) = M_a^{a'}(f_1) T_{a\,+}^{a'\,-}[(g\cdot, b+)] \tag{5.9'}$$

$$\left| \int_b^{b'} f_1(t) g_2(t)\, dt \right| \leqslant M_b^{b'}(f_2) T_{b\,+}^{b'\,-}(g_2) = M_b^{b'}(f_2) T_{b\,+}^{b'\,-}[g(a+, \cdot)]. \tag{5.9''}$$

Moreover

$$\left| \int_a^{a'} \int_b^{b'} f_1(s) f_2(t) g_0\, ds\, dt \right| \leqslant M_a^{a'}(f_1)\, M_b^{b'}(f_2)\, |g(a+, b+)|. \tag{5.10}$$

It should be noted that

$$T_{a\,+}^{a'\,-}(g_1) < \infty \quad T_{b\,+}^{b'\,-}(g_2) < \infty.$$

This follows from the fact that

$$g_1(s) = g^e(s, b) \quad g_2(t) = g^e(a, t)$$

for $s \in (a, a')$ and $t \in (b, b')$, and that g^e satisfies $\hat{F}$ over $\bar{Q}$, since g satisfies $\hat{F}$ over Q. Relation (5.6) follows from equations (5.7) to (5.10).

If g is defined and right continuous at points on the open left and lower edges of $\bar{Q}$ and at (a, b), (5.6) implies that

$$J \leqslant M_a^{a'}(f_1)\, M_b^{b'}(f_2) \,\Big\{ P(Q, g) + T_a^{a'}[g(\cdot, b)] + T_b^{b'}[g(a, \cdot)] + |g(a, b)| \Big\}. \tag{5.11}$$

The 2-dimensional analogue of Cor. 5.1 follows.

COROLLARY 5.2. *Given $f_1 \in L[0, a']$, $f_2 \in L[0, b']$, if ϕ satisfies $\hat{F}$ over $I = \begin{bmatrix} a', & b' \\ 0, & 0 \end{bmatrix}$ and vanishes on the left and lower edges of I, if moreover for some constant A'*

$$P\begin{bmatrix} s, & t \\ 0, & 0 \end{bmatrix}(\phi) \leqslant A' st \quad [(s, t) \in I] \tag{5.12}$$

then for fixed (a', b') and variable (a, b) with $0 < a < a'$, $0 < b < b'$

$$\left| \int_a^{a'} \int_b^{b'} f_1(s) f_2(t) \frac{\phi(s, t)}{s^2 t^2} \, ds dt \right| \leqslant M_o^{a'}(f_1)\, M_a^{b'}(f_2) A' \frac{K}{ab}, \tag{5.13}$$

where K is independent of $M_o^{a'}(f_1) M_o^{b'}(f_2)$ and A'.

Without loss of generality we can assume that $\phi = \phi^{++}$. Cf. § 2. For ϕ^{++} will satisfy (5.12) with ϕ and vanish on the left and lower edges of I with ϕ. Moreover to replace ϕ by ϕ^{++} does not change the left member of (5.13). With $\phi = \phi^{++}$, ϕ is right continuous over $\begin{pmatrix} a', & b' \\ 0 & 0 \end{pmatrix}$ and (5. 11) applies to Q and to the function g with values $g(s, t) = \phi(s, t)/s^2 t^2$ over Q. We evaluate the terms on the right of (5.11) as follows. (The function $g(\cdot, b)$ in (5. 15) has the values $g(s, b)$ for fixed b. The function $g(a, \cdot)$ is similarly defined.)

$$P\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(g) \leqslant A' \frac{K_1}{ab} \qquad \text{[from Cor. 4.3]} \quad (5.14)$$

$$T_a^{a'}[g(\cdot, b)] = T_a^{a'}\left[\frac{\phi(\cdot, b)}{s^2 b^2}\right] \leqslant \frac{A'}{b} \frac{K_2}{a} \quad \text{[from Cor. 4.1]} \quad (5.15)$$

$$T_b^{b'}[g(a, \cdot)] = T_b^{b'}\left[\frac{\phi(a, \cdot)}{a^2 t^2}\right] \leqslant \frac{A'}{a} \frac{K_3}{b} \quad \text{[from Cor. 4.1]} \quad (5.16)$$

$$|g(a, b)| = \left|\frac{\phi(a, b)}{a^2 b^2}\right| \leqslant \frac{A'}{ab} \qquad \text{[from Cor. 5.12].} \quad (4.17)$$

To verify (5.16) observe that

$$T_0^t\left[\frac{\phi(a, \cdot)}{a^2}\right] \leqslant \frac{1}{a^2} P \begin{bmatrix} a, & t \\ 0, & 0 \end{bmatrix} \phi \leqslant \frac{A't}{a} \quad \text{by (5.12)}$$

so that (5.16) follows from Cor. 4.1. Relation (5.15) follows similarly.

A law of the mean for F-variations. A function g with values $g(s, t)$ defined over $Q = U \times V$, where $\bar{Q} = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$ will be said to have *null normal limits on the section* $[t = c]$ of $\bar{Q}$ for $c \in [b, b']$ if

(*i*) $k(u, c+) = 0$

(*ii*) $k(u, c-) = 0 \qquad [u \in U]$

omitting (*i*) if $c = b'$ and (*ii*) if $c = b$.

THEOREM 5.2. *Suppose* $f \in L[b, b']$ *and* $P(Q, k) < \infty$, *where* $Q = U \times V$ *and* $\bar{Q} = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$ *and that* k *has null normal limits on the section* $t = c$ *of* $\bar{Q}$. *If*

$$g(u, v) = \int_q^v f(t) k(u, t) dt \qquad [(u, v) \in Q, q \in V] \qquad (5.18)$$

then for some $\xi \in [b, b']$

$$P(Q, g) = \int_\xi^{b'} |f(t)| \, dt \, P(Q, k) \leqslant M_b^{b'}(|f|) P(Q, k). \quad (5.19)$$

We consider first the case where c is in (b, b') and refer to a partition π of an arbitrary closed subinterval of U and associated set $\mathbf{e}$ as in (3.19). With

$$e_r \Delta_r^{(1)} k(\cdot, t) = \theta(t; \pi, \mathbf{e}) = \theta(t) \qquad (5.20)$$

$$P(Q, g) = \sup T_b^{b'} \int_q^v f(t)\,\theta(t; \pi, e)dt \qquad \text{[as in (3.19)'']}$$

$$= \sup \left[\int_b^c |f(t)|\,|\theta(t)|\,dt + \int_c^{b'} |f(t)|\,|\theta(t)|\,dt \right]$$

$$\leqslant \sup\,[M_b^c(|f|)T_{b+}^{c-}(\theta) + M_c^{b'}(|f|)T_{c+}^{b'-}(\theta)]$$

$$\text{[from (5.2)']}$$

$$\leqslant \sup M_b^{b'}(|f|)T_{b+}^{b'-}(\theta)$$

$$\leqslant M_b^{b'}(|f|)P(Q, k) \qquad \text{[from (3.19)'']} \quad (5.21)$$

from which (5.19) follows. The modification of the proof when $c = b$ or $c = b'$ is obvious.

The next theorem is related to Th. 5.2. It however differs in an essential way from Th. 5.2. We shall refer to the Banach space $M(U)$ of functions measurable over an interval U and bounded almost everywhere in U. The norm of $f \in U$ is

$$\|f\| = \inf_E\,[\sup_{t \in E} |f(t)|], \qquad (5.22)$$

where E is any subset of U with $mE = mU$.

THEOREM 5.3. *Suppose $f \in M(V)$ and $P(Q, k) < \infty$, where $Q = U \times V$. If $k(u, c) = 0$ for $u \in U$ and c fixed in V, and if one sets*

$$g(u, v) = \int_q^v f(t)k(u, t)dt \quad [(u,v) \in Q,\ q \in V]$$

then

$$P(Q, g) \leqslant \|f\| \int_V P(U \times V_t, k)\,dt\,, \qquad (5.23)$$

where $V_t = [c, t]$ or $[t, c]$ according as $c \leqslant t$ or $c > t$.

We refer to a partition π of an arbitrary closed subinterval of U and associated set e as in (3.19)''. With θ formally defined as in (5.20) and taking sups as in (3.19)''

$$P(Q,g) = \sup_{\pi, e} T[V, \lambda]\ \text{[where } \lambda(v) = \int_q^v f(t)\,\theta(t;\pi,e)\,dt]$$

$$= \sup_{\pi, e} \int_V |f(t)\,\theta(t; \pi, e)|\,dt \quad \text{[Cf. def. of } \lambda] \quad (5.24)$$

$$\leqslant \sup_{\pi, e} \|f\| \int_V |\theta(t; \pi, e)|\,dt$$

$$\leqslant \|f\| \int_V T[U, k(\cdot, t\,]\, dt \quad \text{[from def. of } T]$$

$$\leqslant \|f\| \int_V P(U+V_t, k)\, dt \quad \text{[by (2.6)]}$$

taking account of the relation $T(U, k(\cdot, c)) = 0$. This establishes (5.23).

6. **FL-Integrals.** Let $Q = U \times V$ be a general interval in R_2 and let ω map Q into R_1. We shall say that ω is weakly in $L(Q)$ if ω is in L over every closed subinterval of Q. If $Q = \bar{Q}$, a function ω which is weakly in $L(Q)$, is in $L(Q)$ in the ordinary sense. Let (p, q) be any point in Q and set

$$\int_p^u \int_q^v \omega(s, t)\, dsdt = [\bar{\omega}^{(p,q)}(u, v)] \qquad [(u, v) \varepsilon Q] \tag{6.1}$$

for any ω weakly in $L(Q)$. We shall introduce a fundamental subclass of functions ω weakly in $L(Q)$.

DEFINITION. *A function ω weakly in $L(Q)$ will be said to be in $FL(Q)$ if $P(Q, \bar{\omega}) < \infty$ for some indefinite integral $\bar{\omega}$ of ω over Q of the form* (6.1). *The function $\bar{\omega}$ then admits a continuous extension over $\bar{Q}$ and so extended will be termed an FL-integral of ω and again denoted by $\bar{\omega}$.*

A complete justification of this definition requires comment.

Let (p, q) in (6.1) *be called the vertex of $\bar{\omega}^{(p,q)}$. The vertices of a function ω given as weakly in $L(Q)$ will be restricted to Q.*

(I). *If ω is weakly in $L(Q)$, satisfaction of the condition*

$$P[Q, \bar{\omega}^{(p,q)}] < \infty \qquad [(p, q) \varepsilon Q]$$

is independent of the choice of the vertex $(p, q) \varepsilon Q$.

If (p', q') is a second point in Q

$$\bar{\omega}^{(p',q')}(u, v) = \bar{\omega}^{(p,q)}(u, v) - \bar{\omega}(p', v)^{(p,q)} - \bar{\omega}(u, q')^{(p,q)} + \bar{\omega}(p, q')^{(p,q)} \tag{6.2}$$

for $(u, v) \varepsilon Q$, from which it follows that

$$P[Q, \bar{\omega}^{(p,q)}] = P[Q, \bar{\omega}^{(p',q')}] \tag{6.3}$$

thus establishing (I).

(II). *If ω is in $L(Q)$, ω is in $FL(Q)$. If Q is not closed, functions $\omega \varepsilon FL(Q)$ exist with ω not in $L(Q)$, and hence with $|\omega|$ not in $FL(Q)$.*

If ω is in $L(Q)$ and $\bar{\omega}$ is any indefinite integral of ω of the form (6.1) the V-variation

$$V(Q, \bar{\omega}) = \iint_Q |\omega(s, t)|\, dsdt < \infty$$

so that $P(Q, \bar{\omega}) < \infty$. Moreover for any section K of $\bar{Q}$ parallel to an edge of $\bar{Q}$, the function $\bar{\omega} \mid K$ is absolutely continuous. Hence $\bar{\omega}$ in is $\hat{F}(\bar{Q})$. The second statement in (II) is proved by exhibiting an $\omega_0 \epsilon FL(Q)$ not in $L(Q)$, as in MT (7 §6. For it follows from the nature of the differences $\Delta_{rn} g$ used in defining F-variations and V-variations that such differences are never negative for $g = \overline{|\omega|}$. Hence

$$V(Q, \overline{|\omega|}) = P(Q, \overline{|\omega|}). \tag{6.4}$$

If therefore ω_0 is not in $L(Q)$, $|\omega_0|$ is not in $FL(Q)$.

(III). *If ω is in $FL(Q)$, $\bar{\omega}$ is in $\hat{F}(\bar{Q})$. If K is any section of $\bar{Q}$ parallel to an edge of $\bar{Q}$, the function $\bar{\omega} \mid K$ is absolutely continuous over K.*

By virtue of (2.7), $\bar{\omega}$ is in $\hat{F}(\bar{Q})$ since $\bar{\omega} \mid Q$ is in $\hat{F}(Q)$. Hence $\bar{\omega} \mid K$ has a bounded J-variation over K. But over every closed subinterval of K excluding the end points of K, $\bar{\omega} \mid K$ is absolutely continuous, since $\bar{\omega}$ has the form (6.1) over Q. It follows that $\bar{\omega} \mid K$ is absolutely continuous over K.

(IV). *If ω is weakly in $L(Q)$ and admits an integral $\bar{\omega}$ of the form (6.1) over Q with a continuous extension over $\bar{Q}$, then ω need not be in $FL(Q)$.*

This is shown by the function

$$\bar{\omega}(u, v) = (u - \tfrac{1}{2})\left(v \sin \frac{1}{v}\right) \quad \left[(u, v) \varepsilon \begin{pmatrix} 1, & 1 \\ 0, & 0 \end{pmatrix} = Q\right].$$

This function admits a representation of the form (6.1)

with ω weakly in $L(Q)$. Moreover $\bar{\omega}$ admits a continuous extension over $\bar{Q}$. But the functions defined by $\bar{\omega}$ when $u = c \neq \frac{1}{2}$, $0 < c < 1$, are clearly not absolutely continuous. By (III), then ω cannot be in $FL(Q)$.

Given an interval Q and $e > 0$, let Q_e be the closed interval of points of Q whose distance from the boundary of Q is at least e. If $\bar{\omega}$ is an FL-integral over $\bar{Q}$,

$$P(\bar{Q}, \bar{\omega}) = \lim_{e=0} P(Q_e, \bar{\omega}) \qquad \text{[by Cor. 3.6].} \tag{6.5}$$

To evaluate the left member of (6.5) it is sufficient to evaluate $P(I, \bar{\omega})$ for closed subintervals I of Q. The following lemma gives such an evaluation.

LEMMA 6.1. *With ω in L over $I = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$ and $\bar{\omega}$ an integral of ω over I*

$$P(I, \bar{\omega}) = \sup_{\lambda, \mu} \int_a^{a'} \int_b^{b'} \lambda(s) \mu(t) \omega(s, t)\, ds dt \tag{6.6}$$

taking the sup over all subunit pairs (λ, μ) in $C[a, a'] \times C[b, b']$ (or in $C^*[a, a'] \times C^*[b, b']$ Cf. Lemma 2.2.)

Since $\bar{\omega}$ is continuous over I it follows from (2.12) (or from (2.13)) that

$$P(Q, \bar{\omega}) = \sup_{\lambda, \mu} \int_a^{a'} \lambda(s) d_s \int_b^{b'} \mu(t) d_t \bar{\omega}(s, t) \tag{6.7}$$

taking the sup as in the lemma. By virtue of the absolute continuity of $\bar{\omega}(s, \cdot)$ for fixed s this becomes

$$\left.\begin{aligned} &= \sup_{\lambda, \mu} \int_a^{a'} \lambda(s) d_s \int_b^{b'} \mu(t) \left[\int_p^s \omega(\alpha, t) d\alpha \right] dt \qquad \{ p \in [a, a'] \} \\ &= \sup_{\lambda, \mu} \int_a^{a'} \lambda(s) d_s \int_p^s d\alpha \int_b^{b'} \mu(t) \omega(\alpha, t) dt \\ &= \sup_{\lambda, \mu} \int_a^{a'} \lambda(s) \int_b^{b'} \mu(t) \omega(s, t) dt \end{aligned}\right\} \tag{6.7'}$$

thus establishing (6.6).

Convergence in the F-mean. Extending somewhat the classical notion of convergence in the mean suppose that $\omega_n (n = 1, 2, \ldots)$ and ω are weakly in $L(Q)$. We shall say that $\omega_n \to \omega$ in the V-mean over Q if, as $n \to \infty$

$$\iint_Q \left| \omega_n(s, t) - \omega(s, t) \right| dsdt = V[Q, \bar{\omega}_n - \bar{\omega}] \to 0,$$

and that $\omega_n \to \omega$ in the F-mean over Q if

$$P(Q, \bar{\omega}_n - \bar{\omega}) \to 0 \qquad [\text{as } n \to \infty]. \tag{6.8}$$

It is clear that convergence in the V-mean implies convergence in the F-mean, but that convergence in the F-mean does not in general imply convergence in the V-mean. We state the following theorem.

THEOREM 6.1. *If $\omega_n (n = 1, 2, \ldots)$ and ω are in $FL(Q)$, and $\omega_n \to \omega$ in the F-mean over Q then*

$$P(Q, \bar{\omega}_n) \to P(Q, \bar{\omega}) \qquad [\text{as } n \to \infty] \tag{6.9}$$

and if $\bar{\omega}_n$ and $\bar{\omega}$ are FL-integrals with the same vertex then $\bar{\omega}_n(u, v) \to \bar{\omega}(u, v)$ uniformly for $(u, v) \in \bar{Q}$.

We have the relations

$$P(Q, \bar{\omega}) \leqslant P(Q, \bar{\omega}_n) + P(Q, \bar{\omega}_n - \bar{\omega}), \tag{6.10}$$

$$P(Q, \bar{\omega}_n) \leqslant P(Q, \bar{\omega}) + P(Q, \bar{\omega}_n - \bar{\omega}) \tag{6.11}$$

from which (6.9) follows if $\omega_n \to \omega$ in the F-mean over Q. If (u, v) and (p, q) are points in Q and $\bar{\omega}_n$ and $\bar{\omega}$ are FL-integrals with the vertex (p, q)

$$|\bar{\omega}_n(u, v) - \bar{\omega}(u, v)| = \left| \int_p^u \int_q^v (\omega_n(s, t) - \omega(s, t)) dsdt \right|$$
$$\leqslant P(Q, \bar{\omega}_n - \bar{\omega}). \tag{6.12}$$

Relation (6.12) holds also for $(u, v) \in \bar{Q}$ because of the continuity of FL-integrals over $\bar{Q}$. Hence $\bar{\omega}_n(u, v) \to \bar{\omega}(u, v)$ uniformly for $(u, v) \in \bar{Q}$ if $\omega_n \to \omega$ in the F-mean over Q.

COROLLARY 6.1. *If $\omega_n (n = 1, 2, \ldots)$ and ω are in $FL(Q)$, if $\omega_n \to \omega$ pointwise over Q, and $|\omega_n(u, v) - \omega(u, v)|$ is bounded over Q, then $P(Q, \bar{\omega}_n) \to P(Q, \bar{\omega})$ and $\bar{\omega}_n(u, v) \to \bar{\omega}(u, v)$ uniformly over $\bar{Q}$*

Under the hypothesis of the corollary $\omega_n \to \omega$ in the V-mean and hence in the F-mean.

We refer again to the Banach space $M(U)$ of functions f essentially bounded and measurable over U and state a fundamental *factor theorem.*

THEOREM 6.2 *Given* ω *in* $FL(Q)$, *where* $Q = U \times V$, *and* $f_1 \varepsilon M(U)$, $f_1 \varepsilon M(V)$, *the function* $f_1 f_2 \omega$ *with values* $f_1(s) f_2(t) \omega(s, t)$ *satisfies the relation*

$$P(Q, \overline{f_1 f_2 \omega}) \leqslant \|f_1\| \, \|f_2\| \, P(Q, \bar{\omega}), \tag{6.13}$$

where $\|f_1\|$ *and* $\|f_2\|$ *are the norms of* f_1 *and* f_2 *in* $M(U)$ *and* $M(V)$ *respectively.*

We establish the theorem in successive stages of generality.

Case I. f_1 *and* f_2 *continuous.* Let Q_e be the subinterval of Q of points of Q at a distance at least e from the boundary of Q. Suppose $Q_e = U_e \times V_e$. With (λ, μ) subunit elements in $C[U_e] \times C[V_e]$, Lemma 6.1 implies that

$$P(Q_e \, \overline{f_1 f_2 \omega}) = \sup_{\lambda, \mu} \int_{Q_e} \int \lambda(s) f_1(s) \mu(t) f_2(t) \omega(s, t) \, dsdt. \tag{6.14}$$

By operations inverse to those used in deriving (6.7)′ from (6.7) one infers from (6.14) that

$$P(Q_e, \overline{f_1 f_2 \omega}) = \sup_{\lambda, \mu} \int_{U_e} \lambda(s) f_1(s) d_s \int_{V_e} \mu(t) \; f_2(t) d_t \bar{\omega}(s, t) \tag{6.15}$$

so that it follows from a theorem of Fréchet that

$$P(Q_e, \overline{f_1 f_2 \omega}) \leqslant \|f_1\| \, \|f_2\| \, P(Q_e, \bar{\omega}) \tag{6.16}$$

since $\|\lambda\| \leqslant 1$, $\|\mu\| \leqslant 1$. On letting $e \to 0$ in (6.16) we infer the truth of (6.13) in Case I. Use (6.5) and Lemma 2.1.

Case II. The general case. Given $f_1 \varepsilon M(U)$ there exists a sequence of functions $f_1^{(n)}$ continuous over U, such that

$$\|f_1^{(n)}\| \leqslant \|f_1\| \quad \text{[norms in } M(U)],$$

where $f_1^{(n)}$ converges in the mean (in the classical sense) to f_1 as well as pointwise at almost all points in U. Let $f_2^{(n)}$ converge similarly to f_2 over V. Then $f_1^{(n)} f_2^{(n)} \omega$ converges in the V-mean over Q_e to $f_1 f_2 \omega$. As under Case I

$$P(Q_e, \overline{f_1^{(n)} f_2^{(n)} \omega}) \leqslant \|f_1^n\| \; \|f_2^n\| \; P(Q_e \bar{\omega}) \leqslant \|f_1\| \; \|f_2\| \; P(Q, \bar{\omega}). \tag{6.17}$$

As $n \to \infty$ the left member of (6.17) converges to $P(Q_e, \overline{f_1 f_2 g})$ by Th. 6.1. Letting $e \to 0$ we infer the truth of (6.13).

If f with values $f(t)$ is of bounded Jordan variation over $[b, b']$ and if g is in L over $Q = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$ one can write

$$V(Q, \overline{fg}) = \int_a^{a'} \int_b^{b'} | f(t)\, g\,(s, t) \,|\, ds dt \tag{6.19}$$

and can further evaluate this integral by an integration by parts with respect to t. A somewhat more sophisticated procedure is required when one seeks a similar evaluation of $P(Q, \overline{fg})$.

LEMMA 6.2. *Let ω be in $FL(Q)$, where $Q = U \times V$, and f, with values $f(t)$, be of bounded J-variation over V. If* $\bar{Q} = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$

$$P(Q, \overline{f\omega}) \leqslant f\,(b'-)\; P(Q, \bar{\omega}) + \int_{b+}^{b-} P \begin{bmatrix} a', & t \\ a, & b \end{bmatrix} (\bar{\omega}) \;|\, df(t) \,| \tag{6.20}$$

and $f\omega$ is in FL (Q).

We first establish the lemma for the case in which Q is closed. In this case, according to (6.6),

$$P(Q, \overline{f\omega}) = \sup_{\lambda, \mu} \int_b^{b'} \left[\int_a^{a'} \lambda(s)\, \mu\,(t)\, f\,(t)\; \omega\,(s,\, t)\; ds \right] dt \tag{6.21}$$

taking the sup as in (6.6). On integrating by parts with respect to t this becomes the sup of

$$\left[f(t) \int_b^t d\alpha \int_a^{a'} \lambda(s)\mu(\alpha)\omega(s, \alpha)ds\right]_{b+}^{b'-}$$

$$-\int_{b+}^{b'-}\left[\int_b^t d\alpha \int_a^{a'} \lambda(s)\mu(\alpha)\omega(s, \alpha)ds\right]df(t) \quad (6.22)$$

$$= f(b'-) \int_a^{a'} \lambda(s)ds \int_b^{b'} \mu(\alpha)\omega(s, \alpha)d\alpha$$

$$-\int_{b+}^{b'-}\left[\int_a^{a'} \lambda(s)ds \int_b^t \mu(\alpha)\omega(s, \alpha)d\alpha\right]df(t). \quad (6.23)$$

Relation (6.20) follows for Q closed, on taking the required sup of the right member of (6.23) using Lemma 6.1.

In the case in which Q is not closed, (6.20) holds for Q replaced by Q_e with a, a', b, b' replaced by $a+e, a'-e, b+e, b'-e$ respectively. Letting $e \to 0$ (6.20) follows.

To show that f_ω is in $FL(Q)$, one verifies that f_ω is weakly in $L(Q)$ and that $P(Q, \bar{f}_\omega) < \infty$ by (6.20).

Lemma 6.2 leads to the following theorem.

THEOREM 6.3. *If ϕ is in $FL(Q)$, where $Q = U \times V$, if f_1 and f_2 are of bounded J-variation over U and V respectively, and if* $\bar{Q} = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$ *then*

$$P(Q, \overline{f_1 f_2 \phi}) \leqslant |f_1(a'-)|\,|f_2(b'-)|\,P(Q, \bar{\phi})$$

$$+ |f_2(b'-)|\left|\int_{a+}^{a'-} P\begin{bmatrix} s, & b' \\ a, & b \end{bmatrix}(\bar{\phi})\,|df_1(s)|\right.$$

$$+ |f_1(a'-)|\left|\int_{b+}^{b'-} P\begin{bmatrix} a', & t \\ a, & b \end{bmatrix}(\bar{\phi})\,|df_2(t)|\right.$$

$$+\int_{a+}^{a'-} |df_1(s)|\left|\int_{b+}^{b'-} P\begin{bmatrix} s, & t \\ a, & b \end{bmatrix}(\bar{\phi})\,|df_2(t)|\right.. \quad (6.24)$$

This relation is an immediate consequence of the formula obtained from (6.20) on first setting $\omega(s, t) = f_1(s)\,\phi(s, t)$ and $f(t) = f_2(t)$. Thus (6.20) gives

$$P(Q, \overline{f_1 f_2 \phi}) \leqslant f_2(b'-)\, P(Q, \overline{f_1 \phi})$$

$$+\int_{b+}^{b'-} P\begin{bmatrix} a', & t \\ a, & b \end{bmatrix}(\overline{f_1 \phi})\,|df_2(t)|. \quad (6.25)$$

One then uses (6.20) again to obtain the formula

$$P\begin{bmatrix} a', t \\ a, b \end{bmatrix} (\overline{f_1 \phi}) \leqslant f_1(a'-) \; P\begin{bmatrix} a', t \\ a, b \end{bmatrix} (\bar{\phi}) + \int_{a+}^{a'-} P\begin{bmatrix} s, t \\ a, b \end{bmatrix} (\bar{\phi}) \, | \, df_1(s) \, |. \quad (6.26)$$

In particular (6.26), with $t = b'$, will give a replacement for $P(Q, \overline{f_1 \phi})$ in (6.25). One also substitutes the right member of (6.26) in the integral in (6.25). With these substitutions (6.25) yields (6.24).

The following corollary is useful in the Fourier series theory.

Corollary 6.2. *If* ϕ *is in* L *over* $\begin{bmatrix} \pi, \pi \\ 0, 0 \end{bmatrix}$ *and if*

$$P\begin{bmatrix} x, y \\ 0, 0 \end{bmatrix} (\bar{\phi}) \leqslant A \, xy \qquad (0 < x \leqslant \pi) \qquad (0 < y \leqslant \pi) \quad (6.27)$$

for some constant A, *then for*

$$g(s, t) = \frac{\phi(s, t)}{s^2 t^2} \quad (0 < s \leqslant \pi) \quad (0 < t \leqslant \pi) \quad (6.28)$$

and fixed k *with* $(0 < kx < \pi)$ $(0 < ky < \pi)$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left\{ xy \, P\begin{bmatrix} \pi, & \pi \\ kx, & ky \end{bmatrix} (\bar{g}) \right\} \leqslant \frac{4A}{k^2}. \quad (6.29)$$

One applies (6 24) with $f_1(s) = s^{-2}$, $f_2(t) = t^{-2}$, and $Q = \begin{bmatrix} \pi, & \pi \\ kx, & ky \end{bmatrix}$. On the right of (6.24) $f_1(a'-) = f_2(b'-) = \pi^{-2}$, and the respective terms are at most

$$\frac{A}{\pi^2}, \; \frac{2A}{k\pi x}, \; \frac{2A}{k\pi y}, \; \frac{4A}{k^2 xy},$$

taking account of the fact that $s \geqslant kx$ and $t \geqslant ky$ in the integrations involved. Relation (6.29) follows.

7. **Products** $f_1(s) \, f_2(t) \, \hat{g}(s, t)$ **with** $\hat{g}$ **an integral.** Let I be the open interval $\begin{pmatrix} \pi, \pi \\ 0, 0 \end{pmatrix}$. Set

$$\hat{g}(s, t) = \int_0^s \int_0^t g(\alpha, \beta) \, d\alpha \, d\beta \qquad [(s, t) \in I] \quad (7.1)$$

The basic hypotheses will be that g is in L over I, f_1, f_2 continuous and monotone decreasing, with $f_1(s) > 0$, $f_2(t) > 0$, over $[0, \pi]$, and that

$$P(I, \overline{f_1 f_2 g}) < \infty. \tag{7.2}$$

A first conclusion will be that $P(I, f_1 f_2 \hat{g}) < \infty$.

The product $f_1 f_2 g$ will not in general be in L over I and f_1 and f_2 will not in general be bounded. Lemma 7.3 MT (7) shows that there exists a function g in $L(\bar{I})$ such that g/uv is not in $L(\bar{I})$ while $P\left(I, \overline{(\frac{1}{uv} g)}\right) < \infty$.

We begin with somewhat more restrictive hypotheses on f_1 and f_2. Set $I_e = \begin{pmatrix} \pi, \pi \\ e, e \end{pmatrix}$, $\qquad (e > 0)$.

Lemma 7.1. *If f_1 and f_2 are absolutely continuous over $[e, \pi]$, and g is in L over $I = \begin{pmatrix} \pi, \pi \\ 0, 0 \end{pmatrix}$ then*

$$P(I_e, f_1 f_2 \hat{g}) \leqslant P(I_e, \overline{f_1 f_2 g}) + P(I_e, \overline{\dot{f}_1 f_2 \hat{g}_t}) + P(I_e, \underline{f_1 \dot{f}_2 \hat{g}_s}) + P(I_e, \overline{\dot{f}_1 \dot{f}_2 \hat{g}}). \tag{7.3}$$

We first show that the functions

$$f_1 f_2 g, \dot{f}_1 f_2 \hat{g}_t, f_1 \dot{f}_2 \hat{g}_s, \dot{f}_1 \dot{f}_2 \hat{g} \tag{7.4}$$

are in L over I_e. This is clear for $f_1 f_2 g$, since g is in L over I_e while f_1 and f_2 are continuous. Similarly $\dot{f}_1 \dot{f}_2 \hat{g}$ is in L over I_e, since $\dot{f}_1 \dot{f}_2$ is in L over I_e and $\hat{g}$ continuous. Since f_2 is continuous $\dot{f}_1 f_2 \hat{g}_t$ is in L with $\dot{f}_1 \hat{g}_t$. To continue, set

$$\phi(t) = \int_0^\pi |g(\alpha, t)| \, d\alpha.$$

Then ϕ is integrable over $[0, \pi]$. Moreover

$$|\dot{f}_1(s) \hat{g}_t(s, t)| = \dot{f}_1(s) \left|\int_0^s g(\alpha, t) \, d\alpha\right| \leqslant |\dot{f}_1(s) \phi(t)|$$

almost everywhere in I_e, so that $\dot{f}_1 \hat{g}_t$ and hence $\dot{f}_1 f_2 \hat{g}_t$ is in L over I_e. Similarly $f_1 \dot{f}_2 \hat{g}_s$ is in L over I_e.

We note also that $f_1 f_2 \hat{g}$ is of bounded F-variation over I_e by virtue of Lemma 4 2. Since $f_1 f_2 \hat{g}$ is continuous over I_e it follows from (2.12) that

$$P(I_e, f_1 f_2 \hat{g}) = \sup_{\lambda, \mu} J(\lambda, \mu), \tag{7.5}$$

where λ, μ are subunit elements in $C[e,\pi]$, and

$$J(\lambda, \mu) = \int_e^\pi \lambda(s)\, d_s \int_e^\pi \mu(t)\, d_t\, [f_1(s) f_2(t) \hat{g}(s, t)].$$

To reduce $J(\lambda, \mu)$ observe that $f_2 \hat{g}(s, \cdot)$ is absolutely continuous over $[e, \pi]$ for each $s \in [e, \pi]$, and that

$$\int_0^\pi \mu(t)\, d_t\, [f_2(t) \hat{g}(s, t)]$$

$$= \int_0^\pi \mu(t) [\hat{g}_t(s, t)\, f_2(t) + \hat{g}(s, t) \dot{f}_2(t)]\, dt \tag{7.6}$$

$$= \int_0^s d\alpha \int_e^\pi \mu(t)\, [\dot{f}_2(t)\, g(\alpha, t) + \dot{f}_2\, \hat{g}_s(\alpha, t)]\, dt = H(s) \tag{7.7}$$

introducing $H(s)$. It is clear that H is absolutely continuous over $[e\ \pi]$ and that for almost all $s \in [e, \pi]$

$$\dot{H}(s) = \int_e^\pi \mu(t)\, [f_2(t)\, g(s, t) + \dot{f}_2(t)\, \hat{g}_s(s, t)]\, dt. \tag{7.8}$$

Finally

$$J(\lambda, \mu) = \int_e^\pi \lambda(s)\, d_s\, [f_1(s)\, H(s)]$$

$$= \int_e^\pi \lambda(s)\, [\dot{f}_1(s)\, H(s) + f_1(s)\, \dot{H}(s)]\, ds. \tag{7.9}$$

If one us,s $\dot{H}(s)$ from (7.8), and $H(s)$ from (7.6), (7.9) givcs

$$J(\lambda, \mu) = \int_e^\pi \int_e^\pi \lambda(s)\, \mu(t)\, \omega(s, t)\, ds dt, \tag{7.10}$$

where ω is the sum of the four functions appearing in (7.4). Taking the sup of the members of (7.10) over the appropriate (λ, μ)

$$P(I_e, f_1 f_2 \hat{g}) = P(I_e, \bar{\omega})$$

and (7 3) follows.

Uncer the hypotheses in Lemma 7.2 the functions listed in (7.4) are not in general in L over I, although

their integrals will be shown to have bounded F-variations with $\overline{f_1 f_2 g}$.

LEMMA 7.2. *If f_1 and f_2 are strictly decreasing and absolutely continuous over $[e, \pi]$ for every $e > 0$, with values $f_1(s) > 0$ and $f_2(t) > 0$, and if g is in L over $I = \begin{pmatrix} \pi, & \pi \\ 0, & 0 \end{pmatrix}$ then*

$$P(I, f_1 f_2 \hat{g}) \leqslant 4\, P(I, \overline{f_1 f_2 g}). \tag{7.11}$$

We shall proceed under the assumption that

$$P(I, \overline{f_1 f_2 g}) < \infty \tag{7.12}$$

since (7.11) is trivial if (7.12) does not hold. We shall first show that if ϕ is any one of the last three functions listed in (7.4) then

$$P(I_e, \bar{\phi}) \leqslant P(I, \overline{f_1 f_2 g}), \ (e > 0). \tag{7.13}$$

It will then follow from Lemma 7.1 that

$$P(I_e, f_1 f_2 \hat{g}) \leqslant 4\, P(I, \overline{f_1 f_2 g}). \tag{7.14}$$

Relation (7.11) will be implied by (7.14) on letting $e \to 0$.

The case $\phi = f_1 \dot{f}_2 \hat{g}_s$. Recall that $P(I_e, \bar{\phi})$ is the sup for subunit λ, μ in $C[e, \pi]$ of the integral

$$M_e(\lambda, \mu) = \int_e^\pi ds \int_e^\pi \lambda(s)\, \mu(t)\, f_1(s)\, \dot{f}_2(t)\, \hat{g}_s(s, t)\, dt. \tag{7.15}$$

If $K_0^t(\beta)$ gives the values of the characteristic function of the interval $[0, t]$, then for s in a set E_1 with $mE_1 = \pi$, (cf. Fubini),

$$\hat{g}_s(s, t) = \int_0^t g(s, \beta)\, d\beta = \int_0^\pi K_0^t(\beta)\, g(s, \beta)\, d\beta.$$

Substituting for $\hat{g}_s(s, t)$ in (7.15) and interchanging the order of integration with respect to t and β in accordance with Fubini's theorem,

$$M_e(\lambda, \mu) = \int_e^\pi \lambda(s)\, f_1(s)\, ds \int_0^\pi g(s, \beta)\, d\beta \int_e^\pi K_0^t(\beta)\, \mu(t)\, \dot{f}_2(t)\, dt$$
$$(e > 0). \tag{7.16}$$

For $\beta \geqslant e$

$$\int_e^\pi K_o^t(\beta)\,\mu(t)\,\dot{f}_2(t)\;dt = \int_\beta^\pi \mu(t)\,\dot{f}_2(t)\;dt$$

$$= \zeta(\beta)\,[f_2(\pi) - f_2(\beta)] \qquad (7.17)$$

thereby defining $\zeta(\beta)$ for $e \leqslant \beta < \pi$. If one sets $\zeta(\pi) = \mu(\pi)$ an application of the first law of the mean shows that ζ is continuous at $\beta = \pi$, as well as for $\beta \varepsilon [e, \pi)$ and $|\zeta(\beta)| \leqslant 1$. For $0 \leqslant \beta < e$

$$\int_e^\pi K_o^t(\beta)\,\mu(t)\,\dot{f}_2(t)\,dt = \int_c^\pi \mu(t)\,\dot{f}_2(t)\;dt = \zeta(e)\;(f_2(\pi) - f_2(e)).$$

We set $\zeta(\beta) = \zeta(e)$ for $0 \leqslant \beta < e$ and

$$f_2(\pi) - f_2(\beta) = F_e(\beta) \qquad (e \leqslant \beta \leqslant \pi)$$

$$f_2(\pi) - f_2(e) = F_c(\beta) \qquad (0 \leqslant \beta < e).$$

Then (7.16) takes the form

$$M_e(\lambda, \mu) = \int_e^\pi \int_0^\pi \lambda(s)\;\zeta(\beta)\,f_1(s)\;F_e(\beta)\;g(s, \beta)\;ds\,d\beta$$

$$\leqslant P(I, \overline{f_1\,F_e\,g}).$$

Since $|f_2(\beta)| \geqslant |F_e(\beta)|$ it follows from Th. 6.2 that

$$P(I, \overline{f_1 f_2\,g}) \geqslant P(I, \overline{f_1\,F_c\,g}) \geqslant M_e(\lambda, \mu). \qquad (7.18)$$

Relation (7.13) then follows for $\phi = f_1 \dot{f}_2 \hat{g}_s$.

The proof of (7.13) when $\phi = \dot{f}_1 f_2 \hat{g}_t$ is similar. In the case where $\phi = \dot{f}_1 \dot{f}_2 \hat{g}$ one uses the relation

$$\hat{g}(s, t) = \int_0^\pi K_0^s(\alpha)d\alpha \int_0^\pi K_0^t(\beta)g(\alpha, \beta)d\alpha d\beta$$

and makes the corresponding modifications in the proof. Relations (7.13) then lead to (7.14) and (7.11) as stated.

THEOREM 7.1. *If f_1 and f_2 are monotone decreasing and continuous over* $(0, \pi]$ *with values* $f_1(s) > 0$ *and* $f_2(t) > 0$, *and if g is in L over* $I = \binom{\pi,\ \pi}{0,\ 0}$ *then*

$$P(I, \widehat{f_1 f_2 g}) \leqslant 4P(I, \overline{f_1 f_2 g}). \qquad (7.19)$$

It is easy to construct a sequence of strictly decreasing functions $f_1^{(n)}$, absolutely continuous over $[e, \pi]$ for each $e > 0$, with values $0 < f_1^{(n)}(s) < f_1(s)$, and such that

$f_1^{(n)}$ converges pointwise to f_1 over $(0, \pi]$. Let $f_2^{(n)}$ converge similarly to f_2. For each integer n

$$P(I, f_1^{(n)} f_2^{(n)} \hat{g}) \leqslant 4P(I, \overline{f_1^{(n)} f_2^{(n)} g}) \leqslant 4P(I, \overline{f_1 f_2 g}) \qquad (7.20)$$

by Lemma 7.2 and Th. 6.2. On account of the lower semi-continuity of the F-variation with respect to pointwise convergence, (7.20) implies (7.12).

Th. 7.1 will be used to show that our Dini condition implies our de la Vallée Poussin condition.

The F-variation of the mean $\hat{g}/st$. We begin by recalling a law for a function f of bounded J-variation over $[0, a']$. As above set

$$\hat{f}(s) = \int_0^s f(\alpha)\, d\alpha \qquad [0 \leqslant s \leqslant a'].$$

Then, (cf. Titchmarsh p. 409),

$$T_{0+}^{a'}(\hat{f}/s) \leqslant T_{0+}^{a'}(f). \qquad (7.21)$$

LEMMA 7.3. *Suppose $g \in \hat{F}(Q)$ for $Q = (0, a'] \times (0, b']$. If one sets*

$$\zeta(s, t) = \int_0^t g(s, \beta) d\beta \qquad [(s, t) \in \bar{Q}] \qquad (7.22)$$

then

$$P(Q, \zeta/t) \leqslant P(Q, g). \qquad (7.23)$$

We shall apply (3.19)″ to evaluate $P(Q, \zeta/t)$. Using a partition π of an arbitrary closed subinterval of Q and associated set e, set

$$e_r \triangle_r^{(1)}[g(\cdot, t)] = f(t) = f(t; \pi, \mathrm{e}) \qquad (t > 0)$$

and note that

$$e_r \triangle_r^{(1)}\left[\frac{\zeta(\cdot, t)}{t}\right] = \frac{\hat{f}(t)}{t} \qquad (t > 0).$$

It follows from (7.21) that

$$T^{b'} [\hat{f}/t) \leqslant T_{0+}^{b'}(f)$$

so that (3.19)″ implies (7.23) on taking the sup over admissible π, e.

The principal theorem on the mean follows.

THEOREM 7.2. *Suppose* $g \in \hat{F}(Q)$, *where* $Q = (0, a'] \times (0, b']$. *Then*

$$P(Q, \hat{g}/st) \leqslant P(Q, g). \tag{7.24}$$

The lemma conjugate to Lemma 7.3 can be stated as follows. Suppose $\gamma \in \hat{F}(Q)$. If one sets

$$\eta(s, t) = \int_0^s \gamma(\alpha, t)\,d\alpha \qquad (s, t) \in Q \tag{7.25}$$

then

$$P(Q, \eta/s) \leqslant P(Q, \gamma). \tag{7.26}$$

We now set $\gamma = \zeta/t$, as is permissible since γ is in $\hat{F}(Q)$ by virtue of its definition and (7.23). Then

$$\frac{\eta(s, t)}{s} = \frac{\hat{g}}{st} \qquad (s, t) \in Q$$

and (7.26) and (7.23) give

$$P(Q, \eta/s) \leqslant P(Q, \gamma) = P(Q, \zeta/t) \leqslant P(Q, g)$$

as desired.

The preceding theorem on the mean will be used to show that our Jordan condition implies our de la Vallée Poussin condition.

COROLLARY 7.1. *Suppose* g *is in* $L(Q)$, *where* $Q = (0, a'] \times (0, b']$. *Then*

$$P(Q, \hat{g}/st) \leqslant P(Q, g). \tag{7.27}$$

We suppose $P(Q, g) < \infty$, since otherwise (7.27) holds obviously. The function g' with values

$$g'(s, t) = g(s, t) - g(s, b') - g(a', t) + g(a', b') \quad [(s, t) \in Q]$$

is then in $\hat{F}$ over Q. Applying Th 7.2 we have

$$P(Q, \hat{g}'/st) \leqslant P(Q, g') = P(Q, g). \tag{7.28}$$

But for $s > 0$ and $t > 0$

$$\frac{1}{st}\hat{g}' = \frac{1}{st}\hat{g} - \frac{1}{s}\int_0^s g(\alpha, b')\,d\alpha - \frac{1}{t}\int_0^t g(a', \beta)\,d\beta + g(a', b')$$

so that g'/st and g/st differ by functions of a single variable and hence

$$P(Q, \hat{g}'/st) = P(Q, g/st). \tag{7.29}$$

Inequality (7.27) follows from (7.28) and (7.29) and the corollary is proved.

8. **Some Lebesgue theorems generalized.** Suppose that ω is in $L(Q)$, where $\bar{Q} = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$. With $y > 0$ and $b + y < b'$, Lebesgue has shown that

$$\int_a^{a'} \int_b^{b'-y} | \omega(s, t+y) - \omega(s, t) | \, ds dt = o(1), \tag{8.1}$$

where y is the principal infinitesimal Cf. Hobson Vol. I, p. 626. On setting

$$\omega(s, t+y) - \omega(s, t) = \Delta_y[\omega(s, t)] \tag{8.2}$$

and denoting the functions thereby defined p.p. over $\begin{bmatrix} a', & b'-y \\ a, & b \end{bmatrix}$ by $\Delta_y\omega$ the Lebesgue relation (8.1) can be written in the form

$$V\begin{bmatrix} a', & b'-y \\ a, & b \end{bmatrix} (\overline{\Delta_y \omega}) = o(1). \tag{8.3}$$

Since F-variations are at most V-variations over the same interval, (8.3) implies that

$$P\begin{bmatrix} a', & b'-y \\ a, & b \end{bmatrix} (\overline{\Delta_y \omega}) = o(1). \tag{8.4}$$

when ω is in $L(Q)$. However in the application one needs to know that (8.4) holds when ω is in $FL(Q)$, and not necessarily in $L(Q)$. Thus a new theory of the conditions of Lebesgue type is needed when dealing with FL-integrals.

In the proof of the following theorem a *translation principle* will be used which is relatively obvious once it is stated. Let τ_y denote the *y-translation* of the v-axis which replaces the point v by $v+y$. If V is an interval of the v-axis $\tau_y V$ shall denote the translated interval. If

$Q = U \times V$ is a 2-dimensional interval $\tau_y Q$ shall denote $U \times \tau_y V$. We also use the inverse τ_y^{-1} translation in the same sense. If k is defined over Q with values $k(u, v)$ and $k(u, v-y)$ is also defined over Q, we denote the function with values $k(u, v-y)$ by $\tau_y^{-1}k$. With this understood the translation principle states that

$$P(Q, k) = P(\tau_y Q, \tau_y^{-1}k)$$

and follows directly from the definition of an F-variation.

Let ω be in $FL(Q)$, where $\bar{Q} = \begin{bmatrix} a', b' \\ a, b \end{bmatrix}$. Given $x > 0$ and $y > 0$ with $a+x < a'$, $b+y < y'$ let

$$Q_x = Q \cap \tau_x^{-1} Q, \quad Q_y = Q \cap \tau_y^{-1} Q, \quad Q_{xy} = Q \cap \tau_x^{-1} \tau_y^{-1} Q.$$

We set

$$\omega(s+x, t) - \omega(s, t) = \Delta_x \omega(s, t) \qquad [(s, t) \epsilon Q_x]$$
$$\omega(s, t+y) - \omega(s, t) = \Delta_y \omega(s, t) \qquad [(s, t) \epsilon Q_y]$$

and set

$$\Delta_x[\Delta_y \omega] = \Delta_{xy} \omega = \Delta_y[\Delta_x \omega] \text{ for } (s, t) \epsilon Q_{xy}.$$

It is clear that $\Delta_x \omega$, $\Delta_y \omega$ and $\Delta_{xy} \omega$ are weakly in L over Q_x, Q_y, Q_{xy} respectively, and we understand that the integrals $\overline{\Delta_x \omega}$, $\overline{\Delta_y \omega}$, $\overline{\Delta_{xy} \omega}$ are taken with vertices (p, q) in Q_x, Q_y, Q_{xy} respectively. The vertices of $\bar{\omega}$ as stated previously are taken in Q. Observe that for almost all $(u, v) \epsilon Q$,

$$\frac{\partial \bar{\omega}}{\partial u} = \int_q^v \omega(u, t)\,dt \qquad [(p, q) \epsilon Q]$$

and that this partial derivative is weakly in L over Q. With these conventions the principal theorem on FL-integrals generalizing Lebesgue's theorem on L-integrals is as follows.

THEOREM 8.1. *Let ω be in $FL(Q)$, where Q is a general interval with $\bar{Q} = \begin{bmatrix} a', b' \\ a, b \end{bmatrix}$ with $x > 0$, $y > 0$, $a+x < a'$, $b+y < b'$. Then*

$$P\begin{bmatrix} a', b'-y \\ a, b \end{bmatrix}(\overline{\Delta_y \omega}) = o(1) \qquad (8.5)'$$

$$P\begin{bmatrix} a' & x; \ b'-y \\ a & , \ b \end{bmatrix}(\overline{\triangle_{xy}\omega}) = o(1) \tag{8.5}''$$

$$P\begin{bmatrix} a'-x, & b' \\ a & , \ b \end{bmatrix}\left\{ \overline{\triangle_x \frac{c\bar{\omega}}{\partial u}} \right\} \leqslant (b'-b)\, P\begin{bmatrix} a'-x, & b' \\ a & , \ b \end{bmatrix}(\overline{\triangle_x\omega}). \tag{8.6}$$

No generality will be lost in the proof of the theorem if we suppose that $\omega(u, v) = 0$ for $u > a'$, or $v > b'$. This convention in the proof in no way limits the applicability of the theorem.

PROOF OF (8.5)′. By virtue of the convention that $\omega(u, v) = 0$ for $u > a'$ or $v > b'$, $\triangle_y\omega$ is defined for $(u, v)\epsilon Q$ and is in $FL(Q)$ with ω. Let $e > 0$ be so small that $a+e < a'$, $b+e < b'$. Then

$$P(Q, \{\overline{\triangle_y\omega}\} \leqslant P\begin{bmatrix} a+e, & b' \\ a & , \ b \end{bmatrix}(\overline{\triangle_y\omega}) + P\begin{bmatrix} a', & b+e \\ a, & b \end{bmatrix}(\overline{\triangle_y\omega})$$

$$+P\begin{bmatrix} a' & , \ b' \\ a'-e, & b \end{bmatrix}(\overline{\triangle_y\omega}) + P\begin{bmatrix} a', & b' \\ a, & b'-e \end{bmatrix}(\overline{\triangle_y\omega}) + P\begin{bmatrix} a'-e, & b'-e \\ a+e, & b+e \end{bmatrix}(\overline{\triangle_y\omega}) \tag{8.7}$$

Note first that for $(u, v)\epsilon Q$

$$\triangle_y\bar{\omega}(u, v) = \triangle_y\int_p^u ds\int_q^v \omega(s, t)dt = \int_p^u ds\int_q^v \triangle_y[\omega(s, t)]dt$$

$$+\int_p^u ds\int_q^{q+y}\omega(s, t)dt$$

so that $\triangle_y\bar{\omega}$ and $\overline{\triangle_y\omega}$ differ by a function of u and so have the same F-variation over any subinterval of Q. Thus the first term on the right of (8.7) becomes

$$P\begin{bmatrix} a+e, & b' \\ a & , \ b \end{bmatrix}(\triangle_y\bar{\omega}) \leqslant P\begin{bmatrix} a+e, & b' \\ a & , \ b+y \end{bmatrix}(\bar{\omega}) + P\begin{bmatrix} a+e, & b' \\ a & , \ b \end{bmatrix}(\bar{\omega}) \tag{8.8}$$

using an inverse y-translation to obtain the first term on the right of (8.8) and the fact that $\omega(u, v) = 0$ for $v > b'$. Relations similar to (8.8) hold for each of the first four terms on the right of (8.7). Given $\eta > 0$ there exists an $e_1 > 0$ so small that each term on the right of (8.8) is less than η when $e = e_1$. [Cor. 3.5.]. More generally we can suppose that when $e = e_1$ the sum of the first four terms on the right of (8.7) is less than 8η. With $e = e_1$ fixed, the

last term in (8.7) is less than η according to the above theorem of Lebesgue, provided $0 < y < \delta_e$, and δ_e is sufficien ly small. Hence the left number of (8 7) is less than 9η if $0 < y < \delta_e$. This estε blishes (8.5)′.

PROOF OF (8 5)″. The prcof of (8 5)″ is similar to that of (8.5)′ observing first that $\overline{\Delta_{xy}\omega}$ differs from $\Delta_{x_1}\bar{\omega}$ by func ions of u and v as abɔve. One writes $\overline{\Delta_{xy}\omega} = \overline{\Delta_y(\Delta_x\omega)}$ and uses (8 7) with $\Delta_x\omega$ replacing ω, and then further reducεs the resulting terms by u ing (8.7)*. (Given a relation (X) suι h as (8 7) we shall let $(X)^*$ denote the relation obtained from (X) by inter hanging tℎe roles of x and y, u and v, and the associated limits \\\ c term $(X)^*$ conjugate to (X).) Cor. 3.5 and the Lebesgue theorem lead to (8 5)″.

PROOF OF (8.6). For $(u, v)\epsilon Q$,

$$\overline{\Delta_x\frac{\partial\bar{\omega}}{\partial u}}(u, v) = \int_p^u ds\int_q^v dt\int_q^t \Delta_x\omega(s, \beta)d\beta = \int_q^v \overline{\Delta_t\omega}(u, t)dt.$$

It follows from Th. 5.3 with $f(t) \equiv 1$ and $c = q$ therein that (8.6) holds.

The relations (8.5)′* and (8.6)* conjugate to (8.5)′ and (8.6) respecti ely obviously hold.

Lebesgue also introduced the condition

$$\int_x^{\pi-x}\left|\frac{f(v+y)-f(v)}{v}\right|dv = \int_x^{\pi-x}\left|\frac{1}{v}\Delta_y f(v)\right|dv = o(1) \qquad (f\epsilon L[0, \pi])$$

in his test for the convergence of the Fourier series of f. In the corresponding classical 2-dimensional theory one is led to consider a function $\omega\epsilon L(I)$, where $I = \begin{pmatrix}\pi, & \pi\\ 0, & 0\end{pmatrix}$, and integrals such as

$$\int_y^{\pi-y}\left|\frac{1}{v}\Delta_y\omega_1(u, v)\right|dv, \tag{8.9}$$

where

$$\omega_1(u, v) = \int_q^v \omega(u, t)dt = \frac{\partial\bar{\omega}}{\partial u}(u, v).$$

The assumption that ω is in $L(I)$ is again too restrictive for our purposes. To obtain the desired theorems we shall extend a formula due to Hardy, p. 153.

LEMMA 8.1. *If* $0 < 3y < b'$ *and* f *has a bounded* J*-variation over* $(0, b')$ *then*

$$J = \int_y^{b'-y} \left| \frac{f(v+y)-f(v)}{v} \right| dv \leqslant \log 2\, T_y^{b'}(f). \qquad (8.10)$$

Following Hardy's general procedure

$$J \leqslant \int_y^{b'-y} \frac{\phi(v+y)-\phi(v)}{v} dv \qquad [\text{where } \phi(v) = T_{0+}^{v}(f)]\text{.}$$

$$= \Big\{\log v\,[\phi(v+y)-\phi(v)]\Big\}_y^{b'-y} - \int_y^{b'-y} \log v\, d[\phi(v+y)-\phi(v)]$$

$$= \log (b'-y) \int_{b'-y}^{b'} d\phi(v) - \log y \int_y^{2y} d\phi(v)$$

$$- \int_{2y}^{b'} \log (v-y)\, d\phi(v) + \int_y^{b'-y} \log v\, d\phi(v)$$

$$= \int_y^{2y} \log \left(\frac{v}{y}\right) d\phi(v) + \int_{2y}^{b'-y} \log \left(\frac{v}{v-y}\right) d\phi(v)$$

$$+ \int_{b'-y}^{b'} \log \left(\frac{b'-y}{v-y}\right) d\phi(v).$$

The logarithmic factors in these integrals are bounded by log 2. Hence

$$J \leqslant \log 2 \int_y^{b'} d\phi(u) \leqslant \log 2\, T_y^{b'}(f). \qquad (8.11)$$

With ω in $FL(Q)$ and $Q = \begin{bmatrix} a', & b' \\ a+, & 0+ \end{bmatrix}$ set

$$\frac{1}{v} \Delta_y \frac{\partial \bar{\omega}}{\partial u}(u, v) = \theta_y(u, v) \left\{ (u, v) \epsilon Q_y = \begin{bmatrix} a', & b'-y \\ a+, & 0+ \end{bmatrix} \right\}. \qquad (8.12)^{\circ}$$

The function θ_y is weakly L over Q_y and the vertices (p, q) of $\bar{\theta}_y$ will correspondingly be taken in Q_y. The variation

$$J = P\begin{bmatrix} a', & b'-y \\ a, & y \end{bmatrix}(\bar{\theta}_y) = P\begin{bmatrix} a', & b'-y \\ a, & y \end{bmatrix}\left\{ \overline{\frac{1}{v} \Delta_y \frac{\partial \bar{\omega}}{\partial u}} \right\}$$

is clearly independent of the vertices of $\bar{\theta}_y$ in Q_y, but examples will show that it is not independent of the choice

of the vertices (p, q) of $\bar{\omega}$ or $\partial\bar{\omega}/\partial u$ in Q. However J is independent of q for $(p, q)\epsilon Q$ since a change in q changes $\partial\bar{\omega}/\partial u$ by a function of u alone and so changes θ_y not at all. With this understood it appears that we can evaluate J in its most general form by taking the vertex (p, q) of $\bar{\omega}$ *arbitrarily* in Q_y and taking the vertex of $\bar{\theta}_y$ as this same point (p, q). We shall use this result in proving Th. 8.2.

THEOREM 8.2. *Suppose* $\omega \epsilon FL(Q)$, *where* $Q=\begin{bmatrix} a', b' \\ a+, 0+ \end{bmatrix}$. *Then for* $0 < 3y < b'$

$$J = P\begin{bmatrix} a', b'-y \\ a, y \end{bmatrix}(\bar{\theta}_y) \leqslant \log 2\, P\begin{bmatrix} a', b' \\ a, y \end{bmatrix}(\bar{\omega}), \qquad (8.12)$$

where the vertices of $\bar{\theta}_y$ *are taken in* Q_y *and of* $\bar{\omega}$ *in* Q. *In particular* θ_y *is in* $FL(Q)$ *with* ω.

Taking the vertex of $\bar{\omega}$ and $\bar{\theta}_y$ as a point (p, q) in Q_y, as justified above,

$$\bar{\theta}_y(u, v) = \int^u ds \int_q^v \frac{1}{t}\Delta_y \frac{\partial\bar{\omega}}{\partial u}(s, t)dt = \int_q^v \frac{1}{t}\Delta_y\bar{\omega}(u, t)dt. \quad (8.13)$$

Use will be made of a partition π of the interval $[a, a']$ and associated set e as in (3.19)′. Set

$$e_r\Delta_r^{(1)}\bar{\omega}(\cdot, t) = f(t, \pi, e) = f(t) \qquad t\epsilon(0, b'-y].$$

With a sup over π, e as in (3.19)′, using (8.13)

$$J = \sup T_y^{b'-y} \int_q^v \frac{1}{t}\Delta_y f(t; \pi, e)dt$$

$$= \sup \int_y^{b'-y} \left| \frac{1}{t}\Delta_y f(t; \pi, e) \right| dt$$

$$\leqslant \log 2 \sup T_y^{b'} f(t; \pi, e) \text{ [by Lemma 8.1]}. \quad (8.14)$$

That Lemma 8.1 applies is clear; for f is of bounded J-variation over $(0, b']$ for fixed π, e since $\bar{\omega}$ is in $\hat{F}(Q)$. Finally

$$\sup_{\pi, e} T_y^{b'} f(t; \pi e) = P\begin{bmatrix} a', b' \\ a, y \end{bmatrix}(\bar{\omega})$$

in accordance with (3.19)′, and (8.12) follows from (8.14).

In Th. 8.2 the factor $1/v$ in θ_y precedes Δ_y. In Th. 8.3 we shall be concerned with a similar function ζ_y with values

$$\zeta_y(u, v) = \Delta_y \left[\frac{1}{v} \frac{\partial \bar{\omega}}{\partial u}(u, v) \right] \quad \text{for } (u, v) \in Q_y = \begin{bmatrix} a', & b'-y \\ a+, & 0+ \end{bmatrix}$$

in which the factor $1/v$ follows Δ_y. The function ζ_y is weakly in L over Q_y and the vertices of integrals $\bar{\zeta}_y$ shall accordingly be taken in Q_y. The function ζ_y involves $\bar{\omega}$ with its own vertices in $Q \supset Q_y$.

THEOREM 8.3. *Suppose* $\omega \in FL(Q)$ *with* Q *and* Q_y *as in* Th. 8.2. *Then for* $0 < 3y < b'$

$$P(I, \bar{\zeta}_y) \leqslant \log 4 \; P(Q, \bar{\omega}), \quad \left\{ \text{where } I = \begin{bmatrix} a', & b'-y \\ a+, & y \end{bmatrix} \right\} \tag{8.15}$$

where the vertices of $\bar{\zeta}_y$ *are in* Q_y, *and of* $\bar{\omega}$ *in* Q.

We shall establish (8.15) with the aid of (8.12).° Referring to (8.12)° define ϕ over I by setting

$$\phi(u, v) = \theta_y(u, v) - \zeta_y(u, v) = \frac{y \frac{\partial \bar{\omega}}{\partial u}(u, v+y)}{v(v+y)} \quad [(u, v) \in I]. \tag{8.16}$$

From (8.16) and (8.12) it follows that

$$P(I, \bar{\zeta}_y) \leqslant P(I, \bar{\theta}_y) + P(I, \bar{\phi}) \leqslant \log 2 \; P(Q, \bar{\omega}) + P(I, \bar{\phi}) \tag{8.17}$$

and (8.16) will follow on showing that

$$P(I, \bar{\phi}) \leqslant \log 2 \; P(Q, \bar{\omega}). \tag{8.18}$$

PROOF OF (8.18). On setting $\phi(u, v) = 0$ for (u, v) not in I, the vertex of the extended $\bar{\phi}$ in (8.18) can be taken arbitrarily and in particular as identical with (p, q) arbitrarily chosen in Q as the vertex of $\bar{\omega}$. Let K_y be the characteristic function of $[y, b'-y]$. We then have

$$\begin{aligned} \bar{\phi}(u, v) &= \int_p^u ds \int_q^v \frac{K_y(t) y \frac{\partial \omega}{\partial u}(s, t+y)}{t(t+y)} dt \\ &= \int_q^v f_y(t) \bar{\omega}(u, t+y) dt \quad \left[\text{setting } \frac{K_y(t) y}{t(t+y)} = f_y(t)\right]. \end{aligned} \tag{8.19}$$

Th. 5.2 will now be applied with $\tau_y^{-1}Q$ as the interval $U \times V$ of Th. 5.2, with $f(t) = f_y(t)$, $k(u, v) = \bar{\omega}(u, v+y)$ and $c = q-y$. The conditions in Th. 5.2 on $k(u, c+)$ and $k(u, c-)$ are satisfied since k is continuous and $k(u, c) = \bar{\omega}(u, q) = 0$ for $u \in (a, a']$. The function g of Th. 5.2 becomes $\bar{\phi}$ by virtue of (8.19) so that Th. 5 2 implies that

$$P(\tau_y^{-1}Q, \bar{\phi}) \leqslant M_y^{b'-y}(|f_y|)\, P(\tau_y^{-1}Q, k). \qquad (8.20)$$

Relation (8.20) yields (8.18) as we shall now see. From the definition of the extended ϕ and from the translation princi, le respectively,

$$P(\tau_y^{-1}Q, \bar{\phi}) = P(I, \bar{\phi}), \qquad P(\tau_y^{-1}Q, k) = P(Q, \bar{\omega}),$$

while subject to the conditions $0 < 3y < b', y < v < b'-y$

$$M_y^{b'-y}(|f_y|) = \sup_v \left| \int_{b'-y}^{v} f_y(t)dt \right| = \left[\log \frac{v}{v+y} \right]_y^{b'-y} \leqslant \log 2.$$

Thus (8.18) follows from (8.20).

The theorem follows from (8.17).

Since $P(Q, \bar{\omega})$ is finite (8.15) can be written in the form

$$P(\bar{I}, \bar{\zeta}_y) \leqslant \log 4\, P(\bar{Q}, \bar{\omega}) \qquad (8.21)$$

under the conditions of Th. 8 3. In justification of (8.21) recall that $\bar{\omega}$, originally defined over Q, admits a continuous extension over $\bar{Q}$ when $(Q, \bar{\omega})$ is finite. The same is true of $\bar{\zeta}_y$ over $\bar{I}$, and (8 21) then holds in accordance with Lemma 2.1. We are denoting the extensions of $\bar{\omega}$ and $\bar{\zeta}_y$ by the same symbols.

9. **Integrals and F-variations of differences.** We continue with integrals of the differences $\Delta_x g$ and $\Delta_y g$ defined in §9 which arise in various branches of analysis. If f, with values $f(s)$, is in $L[a_0, a']$ and $a_0 < a < a'$ then on setting $x = a'-a$

$$\int_a^{a'} f(s)ds = \int_{a_0}^{a} \Delta_x f(s)ds + \int_{a_0}^{a_0+x} f(s)ds. \qquad (9.1)$$

This is an easy consequence of the relation

$$\int_{a_0}^{a} \Delta_x f(s)ds = \int_{a_0+x}^{a'} f(s)ds - \int_{a_0}^{a} f(s)ds \; [a'-a=x] \tag{9.2}$$

and is of great use in delimiting the integral on the left of (9.1) in terms of conditions on the integrals on the right of (9.1). In stating a 2-dimensional analogue for (9.1) it will be useful to introduce the notation

$$\int_a^{a'} \int_b^{b'} g(s, t)dsdt = \int\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(g)$$

for the L-integral of g over $Q = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$ thereby paralleling the notation $P\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(g)$ for F-variation of g over Q. The 2-dimensional analogue of (9.1) can then be stated as follows.

LEMMA 9.1. *If g is in L over $\begin{bmatrix} a, & b' \\ a_0, & b \end{bmatrix}$ if $a_0 < a < a'$ and if $a'-a = x > 0$ then*

$$\int\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(g) = \int\begin{bmatrix} a, & b' \\ a_0, & b \end{bmatrix}(\Delta_x g) + \int\begin{bmatrix} a_0+x, & b' \\ a_0, & b \end{bmatrix}(g). \tag{9.3}$$

This lemma follows at once from (9.1).

Equation (9.3) leads to a relation involving the F-variation of the FL-integral $\bar{g}$. Cf. §6. By virtue of the definition of the F-variation

$$P\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(\bar{g}) \geqslant |\bar{g}(a', b') - \bar{g}(a', b) - \bar{g}(a, b') + \bar{g}(a, b)| \tag{9.4}$$

The right member of (9.4) is the absolute value of the integral $\int\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(g)$ so that (9.4) yields the elementary but fundamental relation

$$\left|\int\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(g)\right| \leqslant P\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(\bar{g}). \tag{9.5}$$

This relation and the preceding lemma give the following.

LEMMA 9.2. *For the intervals of Lemma 9.1*

$$\left|\int\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(g)\right| \leqslant P\begin{bmatrix} a, & b' \\ a_0, & b \end{bmatrix}(\overline{\Delta_x(g)}) + \left|\int\begin{bmatrix} a_0+x, & b' \\ a_0, & b \end{bmatrix}(g)\right| \tag{9.6}$$

The following analogue of Lemma 9.1 is considerably deeper.

THEOREM 9.1. *If* ω *is in* $FL(Q)$, *where* $Q=\begin{pmatrix} a', & b' \\ a_0, & b \end{pmatrix}$, *if* $a_0 < a < a'$ *and if* $a'-a=x>0$, *then*

$$P\begin{bmatrix} a', & b' \\ a\ , & b \end{bmatrix}(\bar{\omega}) \leqslant P\begin{bmatrix} a\ , & b' \\ a_0, & b \end{bmatrix}(\overline{\Delta_x \omega})+P\begin{bmatrix} a_0+x, & b' \\ a_0\quad , & b \end{bmatrix}(\bar{\omega}). \quad (9.7)$$

We begin with the case in which ω is in $L(Q)$.

Let λ and μ be subunit elements in $C^*[a, a']$ and $C^*[b, b']$. Cf. §2. In accordance with Lemma 6.1, when ω is in $L(Q)$

$$P\begin{bmatrix} a', & b' \\ a\ , & b \end{bmatrix}(\bar{\omega}) = \sup_{\lambda,\mu} \int\begin{bmatrix} a', & b' \\ a\ , & b \end{bmatrix}(\lambda\,\mu\,\omega).$$

Since $\lambda(a)=\lambda(a')$, $\lambda(s)$ can be continuously extended over $[a_0, a']$ so that $\lambda(s+x)=\lambda(s)$ for $s\,\epsilon\,[a_0, a]$. It follows that

$$\Delta_x[\lambda(s)\mu(t)\omega(s, t)] = \lambda(s)\mu(t)\Delta_x\omega(s, t) \qquad (9.8)$$

for $s\,\epsilon[a_0, a]$ and $t\,\epsilon[b, b']$. With λ so extended, (9.3) taken with (9.8), gives

$$\int\begin{bmatrix} a', & b' \\ a\ , & b \end{bmatrix}(\lambda\,\mu\,\omega) = \int\begin{bmatrix} a\ , & b' \\ a_0, & b \end{bmatrix}(\lambda\,\mu\,\Delta_x\omega)+\int\begin{bmatrix} a_0+x, & b' \\ a_0\quad , & b \end{bmatrix}(\lambda\,\mu\,g).$$
$$(9.9)$$

One applies Lemma 6.1 to the terms of (9.9), taking the sup over all subunit (λ, μ) in $C^*[a, a']\times C^*[b, b']$ (with extensions as defined above). It follows that (9.7) holds for $\omega\,\epsilon L(Q)$.

In the general case ω is in $FL(Q)$. If $e>0$ is any sufficiently small constant (9.7) holds with a_0, a', b, b' replaced by a_0+e, $a'-e$, $b+e$, $b'-e$, since ω is in $L(I)$ for $I=\begin{bmatrix} a'-e, & b'-e \\ a_0+e, & b+e \end{bmatrix}$. On letting $e\to 0$ (9.7) follows in accordance with Cor. 3.6.

Equation (9.3) can be written in the form

$$\int\begin{bmatrix} a_0+x, & b' \\ a_0\quad , & b \end{bmatrix}(g) = \int\begin{bmatrix} a', & b' \\ a\ , & b \end{bmatrix}(g) - \int\begin{bmatrix} a\ , & b' \\ a_0, & b \end{bmatrix}(\Delta_x g) \qquad (9.10)$$

with $a_0 < a < a'$ and $a' - a = x$ as previously. One is accordingly led to the following theorem.

THEOREM 9.2. *Under the hypotheses of Th.* 9.1

$$P\begin{bmatrix} a_0+x, & b' \\ a_0 & , b \end{bmatrix}(\bar{\omega}) \leqslant P\begin{bmatrix} a & , b' \\ a_0, & b \end{bmatrix}(\overline{\Delta_x \omega}) + P\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(\bar{\omega}). \quad (9.11)$$

In proving (9.2) one takes λ as a subunit element in $C^*[a_0, a_0+x]$ and extends $\lambda(s)$ over $[a_0, a']$ so that $\lambda(s) = \lambda(s+x)$ for $s \epsilon [a_0, a]$. Otherwise one proceeds essentially as in the proof of Th. 9.1.

The lemmas and theorems conjugate to the above, with interchange of roles of x with y, u with v, obviously hold. Thus

THEOREM 9.1*. *If ω is in $FL(Q)$, where* $Q = \begin{pmatrix} a', & b' \\ a, & b \end{pmatrix}$ *if $b_0 < b < b'$, and if $b' - b = y$, then*

$$P\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(\bar{\omega}) \leqslant P\begin{bmatrix} a', & b \\ a, & b_0 \end{bmatrix}(\overline{\Delta_y \omega}) + P\begin{bmatrix} a', & b_0+y \\ a, & b_0 \end{bmatrix}(\bar{\omega}) \quad (9.12)$$

$$P\begin{bmatrix} a', & b_0+y \\ a, & b_0 \end{bmatrix}(\bar{\omega}) \leqslant P\begin{bmatrix} a', & b \\ a, & b_0 \end{bmatrix}(\overline{\Delta_y \omega}) + P\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(\bar{\omega}). \quad (9.13)$$

From (9.1) one readily derives the following.

THEOREM 9.1a. *If ω is in $L(Q)$, where* $\bar{Q} = \begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}$, *if $b_0 < b < b'$, and if $b' - b = y$, then*

$$V\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(\bar{\omega}) \leqslant V\begin{bmatrix} a', & b \\ a, & b_0 \end{bmatrix}(\overline{\Delta_y \omega}) + V\begin{bmatrix} a', & b_0+y \\ a, & b_0 \end{bmatrix}(\bar{\omega}) \quad (9.12a)$$

$$V\begin{bmatrix} a', & b_0+y \\ a, & b_0 \end{bmatrix}(\bar{\omega}) \leqslant V\begin{bmatrix} a', & b \\ a, & b_0 \end{bmatrix}(\overline{\Delta_y \omega}) + V\begin{bmatrix} a', & b' \\ a, & b \end{bmatrix}(\bar{\omega}). \quad (9.13a)$$

10. **The F-variation of g over sequences of intervals.** Consider a sequence of closed intervals of the form

$$Q_1 \subset Q_2 \subset Q_3 \cdots \quad (10.1)$$

whose union is an interval Q. If Q is closed then for m sufficiently large and $n > m$, $Q = Q_n$. In case Q is not closed let $E^1, \ldots, E^\nu$, $0 < \nu \leqslant 4$, be the edges of $\bar{Q}$ not in Q. Then for N sufficiently large and $n > N$, Q_n will intersect any edge of $\bar{Q}$ not one of the edges $E^1, \ldots, E^\nu$.

In case Q is not closed and $n > N$, as above, let E^i_n be the edge of Q_n nearest E^i, $i = 1, \ldots, \nu$. Let J^i_n be the maximal subinterval of Q which intersects Q_n only in E^i_n. It is clear that

$$Q = Q_n + \text{Union}_i\, J^i_n (i = 1, \ldots, \nu) \tag{10.2}$$

and that the maximum distance of J^i_n from E^i tends to zero as $n \to \infty$.

With this understood the basic theorem is as follows.

THEOREM 10.1. *If the sequence* (10.1) *has Q as a union and g is any function defined over Q*

$$P(Q, g) = \lim_{n \to \infty} P(Q_n, g) \tag{10.3}$$

with (10 3) *holding even when either member of* (10.3) *is infinite.*

If Q is closed, $Q_n = Q$ for n sufficiently large and (10 3) holds trivially.

Suppose then that Q is not closed. We distinguish two cases.

Case I. $P(Q, g) < \infty$. It follows from (10.2) that
$P(Q, g) \leqslant P(Q_n, g) + \sum_i P(J^{(i)}_n, g) \qquad (i = 1, \ldots, \nu)$

and hence

$$P(Q, g) \geqslant P(Q_n, g) \geqslant P(Q, g) - \sum_i P(J^{(i)}_n, g). \tag{10.4}$$

By virtue of Cor. 3.1, $P(J^{(i)}_n, g) \to 0$ as $n \to \infty$ and (10. 3) follows in Case I.

Case II. $P(Q, g) = \infty$. It follows from the definition of the F-variation for a non-closed interval Q, that if N is an arbitrarily large number, $P(Q', g) > N$ for some closed subinterval $Q' \subset Q$. Since Q' is closed $Q_n \supset Q'$ for n sufficiently large, and hence for such n

$$P(Q_n, g) > N.$$

Thus $P(Q_n, g) \uparrow \infty$ as $n \to \infty$. Thus (10.3) holds in this case as well.

Let I_n, $n = 1, 2, \ldots$ be any sequence of non-overlapping closed intervals. Such a sequence will be termed *admissible* if

$$\bigcup_{n=1}^{m} I_n = Q_m \qquad (m = 1, 2, \ldots) \qquad (10.5)$$

is an interval whose intersection with I_{m+1} is an edge common to I_{m+1} and Q_m.

THEOREM 10.2. *If I_n is an admissible sequence of non-overlapping closed intervals whose union is an interval Q, and if g is a function defined over Q*

$$P(Q, g) \leqslant \sum_n P(I_n, g) \qquad (n = 1, 2, \ldots). \qquad (10.6)$$

The intervals Q_m defined in (10.5) form a sequence

$$Q_1 \subset Q_2 \subset Q_3 \subset \cdots$$

whose union is Q. The relations

$$P(Q_m, g) \leqslant \sum_{n=1}^{m} P(I_n, g)$$

and

$$P(Q, g) = \lim_{m \to \infty} P(Q_m, g)$$

imply (10.6).

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SYMBOLIC INVERSIONS OF THE FOURIER SINE TRANSFORM AND OF RELATED TRANSFORMS

By D. V. WIDDER

1. Introduction. In 1947 the author showed [3] how the convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\,\phi(t)\,dt \qquad (1.1)$$

can be inverted by a linear differential operator of infinite order for a restricted class of kernels $G(x)$. In 1949 I. I. Hirschman, Jr. and the author [1] greatly enlarged the class of admissible kernels. A noteworthy one of the latter class is

$$G(x) = e^{-e^{-x}}. \qquad (1.2)$$

For this choice of $G(x)$ the transform (1.1) is inverted by the operator $1/\Gamma(D)$ in the following sense:

$$\frac{1}{\Gamma(s)} = e^{\gamma s}\, s \prod_{1}^{\infty}(1+s/k)e^{-s/k} \qquad (1.3)$$

$$\phi(x) = \lim_{n\to\infty} e^{\gamma D} D \prod_{1}^{n}(1+D/k)e^{-D/k} f(x). \qquad (1.4)$$

Here γ is Euler's constant, D stands for differentiation with respect to x and $e^{aD}f(x)$ is to be interpreted in the usual way as $f(x+a)$. As was pointed out in [1, p. 136], if $f(x) = g(e^{-x})$, $\phi(t) = \psi(e^{t})e^{t}$, equation (1.1) becomes the Laplace transform

$$g(x) = \int_{0}^{\infty} e^{-xt}\,\psi(t)\,dt, \qquad (1.5)$$

and equation (1.4) becomes [4, p. 288]

$$\lim_{n\to\infty} \frac{(-1)^n}{n!}\, g^{(n)}(n/t)\,(n/t)^{n+1} = \psi(t). \qquad (1.6)$$

We shall need this inversion in the present paper.

Observe that the bilateral Laplace transform of the function (1.2) is $\Gamma(s)$, the reciprocal of the entire func-

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tion (1.3) which has real zeros only. This is the characteristic property of the kernels studied in [3] and [1]. In the present paper it is proposed to enlarge the class of admissible kernels still further to include some whose bilateral Laplace transforms are meromorphic functions $M(s)$ with real zeros and poles only and to show that (1.1) is still inverted by $1/M(D)$ in a precise sense. As a special case the Fourier sine transform

$$g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xt \, \psi(t) dt, \tag{1.7}$$

when put in the form (1.1) has $G(x) = \text{sine } e^{-x}$, a function whose bilateral Laplace transform is

$$M(s) = \sqrt{\frac{2}{\pi}} \Gamma(s) \sin \frac{\pi s}{2}. \tag{1.8}$$

The operation $\sqrt{\pi/2} \sin(\pi D/2)$ applied to $f(x)$ will be interpreted as a convolution, and the operation $1/\Gamma(D)$ has already been described in (1.4). We will thus invert (1.7) in two steps, the first of which is an integration, the second, a limiting process (1.6). It is important to observe that we shall make no demand on the local properties of $\psi(t)$ except continuity, whereas the classical inversion of (1.7) requires bounded variation or some comparable property. The self-reciprocal nature of the transform (1.7) will appear anew [2, p. 213] as a result of the equation $M(s)M(1-s) = 1$.

2. **Known results.** In this section we set down the known results from [1] and [3] which we shall need here. To simplify the statement let us introduce a definition.

DEFINITION 1. *The entire function $E(s)$ is admissible if*

$$E(s) = e^{qs} s^r \prod_1^\infty (1 - s/a_k) \, e^{s/a_k}, \tag{2.1}$$

where r is a non-negative integer and q, $a_1, a_2, \ldots$ are real numbers such that $\sum_1^\infty a_k^{-2} < \infty$.

A *corresponding interval* I_E is an interval containing no zero of $E(s)$ except perhaps at an end point.

For example, $E(s)$ might be $1/\Gamma(s)$ with I_E the interval $(-1, 0)$ or $(0, \infty)$. By Theorem 3*a* of [1] we have the following result.

THEOREM A. *If $E(s)$ is admissible with corresponding interval I_E, then*

$$\frac{1}{E(s)} = \int_{-\infty}^{\infty} e^{-st}\, G(t)\,dt \qquad \sigma \varepsilon I_E,\ s = \sigma + i\tau, \tag{2.2}$$

the integral converging absolutely. The kernel $G(t)$ is given explicitly by the absolutely convergent integral

$$G(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{E(s)}\,ds \qquad c\,\varepsilon\, I_E,\ -\infty < t < \infty. \tag{2.3}$$

We now state the inversion result, Theorems 19*b*, 23*b*, 24*d*, 26*c* of [1].

THEOREM B. *If $G(t)$ is defined by* (2.3) *and if $f(x)$ and $\phi(t)$ are related by equation* (1.1), *the integral converging for all x, then*

$$\phi(x) = E(D)f(x) \tag{2.4}$$

at points of continuity x of $\phi(t)$.

3. **The general inversion theorem.** The meromorphic function $M(s)$ is to be the reciprocal of two admissible functions, $M(s) = E(s)/F(s)$. We do not require that the zeros of $E(s)$ should be all distinct from those of $F(s)$. For example, $E(s) = \sin(\pi s/2)$, $F(s) = 1/\Gamma(s)$ as in (1.8). We now make precise the meaning of the operator $1/M(D)$. It shall be the product of the two operators $1/E(D)$, performed first, and $F(D)$. The latter is defined by its infinite product expansion just as the operation (1.4) followed from (1.3). By virtue of equation (2.2) and in view of the usual interpretation of $e^{-tD}f(x)$ as $f(x-t)$ we make the following definition.

DEFINITION 2. *If $E(s)$ is an admissible function satisfying equation* (2.2), *then*

$$\frac{1}{E(D)}f(x) = \int_{-\infty}^{\infty} f(x-t)G(t)\,dt \tag{3.1}$$

whenever this integral converges.

Our principal result can now be stated as follows.

THEOREM 1. *Let $M(s)=E(s)/F(s)$ be the quotient of two admissible functions and let (a, b) be an interval included in corresponding intervals I_E and I_F. Suppose further that $\phi(t)$ is continuous, $-\infty<t<\infty$, and that*

1. $\int_{-\infty}^{\infty}|M(\sigma+i\tau)|\,d\tau<\infty$ *for every σ in $a<\sigma<b$*
2. $\lim_{|\tau|\to\infty} M(\sigma+i\tau)=0$ *uniformly in $a<\sigma<b$*
3. $\phi(t)=O(e^{\alpha t}) \quad t\to-\infty$
$\quad =O(e^{\beta t}) \quad t\to+\infty, \qquad a<\alpha<\beta<b.$

Then there exists a function $K(t)$ sueh that

A. $$M(s)=\int_{-\infty}^{\infty} e^{-st}K(t)dt \qquad a<\sigma<b,$$

the integral converging absolutely. If

B. $$f(x)=\int_{-\infty}^{\infty} K(x-t)\phi(t)dt,$$

this integral converges absolutely, $-\infty<x<\infty$, and

C. $$\frac{1}{M(D)}f(x)=\phi(x) \qquad -\infty<x<\infty.$$

An example of a function $M(s)$ satisfying these conditions is the function (1.8). It has poles at the points $-2, -4, -6, \ldots$ and zeros at $1, 3, 5, \ldots$. Clearly

$$\sin \pi s/2=O(e^{\pi|y|/2}) \qquad |y|\to\infty$$

uniformly in any finite interval. By Stirling's formula,

$$|\Gamma(\sigma+iy)|\sim\sqrt{2\pi}\,e^{-\pi|y|/2}|y|^{\sigma-\frac{1}{2}} \qquad |y|\to\infty,$$

we see that

$$M(\sigma+iy)=O(|y|^{-1-\delta}) \qquad |y|\to\infty$$

uniformly in $-R\leqslant\sigma\leqslant-\frac{1}{2}-\delta$ for any $\delta>0$ and any $R>\frac{1}{2}+\delta$. Hence hypotheses 1 and 2 are satisfied if $a=-1$ and $-1<b<-\frac{1}{2}$.

4. A lemma about convolutions. Before proving Theorem 1 let us discuss the behaviour of the convolution of two functions in the neighborhood of infinity.

DEFINITION 3. *A function $f(x)$, bounded on every finite interval, belongs to the class $\{\alpha, \beta\}$ if*

$$f(x) = O(e^{\alpha x}) \qquad x \to +\infty$$
$$= O(e^{\beta x}) \qquad x \to -\infty.$$

By the use of this notation we now state the desired preliminary result as follows.

LEMMA. *If $g(x)\,\varepsilon\,\{a, b\}$, $h(x)\,\varepsilon\,\{\alpha, \beta\}$, where $a < \beta$ and $\alpha < b$, then*

$$\int_{-\infty}^{\infty} |g(x-t)h(t)|\,dt\,\varepsilon\,\{\text{Max}\,(a, \alpha),\ \text{Min}\,(b, \beta)\}. \quad (4.1)$$

First suppose that $x > 0$. Then the integral (4.1) is

$$e^{\alpha x}\int_{-\infty}^{x} |g(t)|\,O(e^{-\alpha t})dt + e^{\beta x}\int_{x}^{\infty} |g(t|\,O(e^{-\beta t})dt$$

$$= e^{\alpha x}\int_{-\infty}^{0} O(e^{bt-\alpha t})dt + e^{\alpha x}\int_{0}^{x} O(e^{at-\alpha t})dt + e^{\beta x}\int_{x}^{\infty} O(e^{at-\beta t})dt$$

$$= O(e^{\alpha x}) + O(e^{a x}) + O(e^{a x}) + O(e^{a x}).$$

This proves the result in so far as it concerns $x \to +\infty$. The remaining conclusion is reached by a similar proof or by the application of the result just obtained to $g(-x)$ and $h(-x)$.

5. **Proof of Theorem 1.** Conclusion A follows at once from hypotheses 1 and 2 by use of a theorem of Hamburger [4, p. 265]. In fact the classical inversion formula [4, p. 241] gives $K(t)$ explicitly,

$$K(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(s)e^{st}ds \qquad -\infty < t < \infty,\ a < c < b,$$

the integral converging absolutely. Hence

$$|K(t)| \leqslant \frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty} |M(c+iy)|\,dy,$$

from which we see that $K(t)\,\varepsilon\,\{\alpha', \beta'\}$, where $a < \alpha' < \alpha < \beta < \beta' < b$ and α, β are the constants of hypothesis 3. Hence the lemma shows that

$$\int_{-\infty}^{\infty} |K(x-t)\phi(t)|\,dt\,\varepsilon\,\{\beta, \alpha\}. \quad (5.1)$$

This establishes the absolute convergence of the integral of Conclusion B.

We turn now to the final conclusion C. By Theorem A, there exists a function $G(t)$ satisfying equations (2.2) and (2.3). By Definition 2, equation (3.1) will now hold provided that the integral (3.1) converges. Since (a, b), and hence (α, β), is included in I_E, it is clear from equation (2.3) that $G(t) \epsilon \{\alpha-\delta, \beta+\delta\}$ for some $\delta > 0$. A second application of the lemma shows that the integral (3.1) belongs to $\{\beta, \alpha\}$ and as a consequence converges absolutely for all x.

Since (a, b) is included in I_E it follows that the bilateral Laplace transforms of $K(t)$ and of $G(t)$ have a common region of absolute convergence and hence by the product theorem [4, p. 258] that

$$M(s) . \frac{1}{E(s)} = \frac{1}{F(s)} = \int_{-\infty}^{\infty} e^{-st} H(t)dt \quad a < \sigma < b \qquad (5.2)$$

$$H(x) = \int_{-\infty}^{\infty} K(x-t)\ G(t)dt \quad -\infty < x < \infty, \qquad (5.3)$$

both integrals converging absolutely. From equations (3.1) and (5.3) we have

$$\frac{1}{E(D)} f(x) = \int_{-\infty}^{\infty} G(t)dt \int_{-\infty}^{\infty} K(x-t-y)\ \phi(y)dy$$

$$= \int_{-\infty}^{\infty} \phi(y)dy \int_{-\infty}^{\infty} K(x-y-t)\ G(t)dt$$

$$= \int_{-\infty}^{\infty} H(x-y)\ \phi(y)dy. \qquad (5.4)$$

The change in the order of integration here employed is justified by Fubini's theorem. We need only observe that

$$\int_{-\infty}^{\infty} | G(x-t) | dt \int_{-\infty}^{\infty} | K(t-y)\ \phi(y) | dy < \infty.$$

By the use of the lemma this follows from (5.1) and the fact that (α, β) is included in I_E. Finally by Theorem B we may now apply the operator $F(D)$ to equation (5.4), and obtain

$$F(D) . \frac{1}{E(D)} f(x) = \frac{1}{M(D)} f(x) = \phi(x) \qquad -\infty < x < \infty.$$

This completes the proof of the theorem.

6. **Reciprocal transforms.** We prove next a companion result to Theorem 1, which when applied to the Fourier sine transform, for example, will reveal its self-reciprocal character.

THEOREM 2. *Let $M(s)$ and $\phi(t)$ satisfy the conditions of Theorem* 1 *and in addition let*

4. $M(s)\ M(1-s)=1 \qquad a<\sigma<b, \quad (1-b,\ 1-a)\subset I_E.$

Then if $f(x)$, defined as in Theorem 1, *satisfies the conditions*

5. $$\begin{aligned} f(x) &= O(e^{\gamma x}) \quad x\to-\infty \\ &\ O(e^{\delta x}) \quad x\to+\infty \qquad 1-b<\gamma<\delta<1-a, \end{aligned}$$

we have

$$\phi(x)=\int_{-\infty}^{\infty}K(t-x)\ e^{x-t}f(t)dt \quad -\infty<x<\infty,$$

the integral converging absolutely.

By Conclusion A of Theorem 1

$$M(1-s)=\int_{-\infty}^{\infty}e^{-st}\ e^{t}\ K(-t)dt \qquad 1-b<\sigma<1-a. \quad (6.1)$$

Since $(1-b,\ 1-a)$ is included in I_F, the integrals (5.2) and (6.1) have a common region of absolute convergence. Again using the product theorem we have, by virtue of hypothesis 4,

$$\frac{1}{E(s)}=\frac{1}{F(s)}\cdot M(1-s)$$

$$G(x)=\int_{-\infty}^{\infty}H(t)\ e^{x-t}K(t-x)dt \qquad -\infty<x<\infty,$$

the integral converging absolutely. Putting this value of $G(x)$ in equation (3.1) we have

$$\begin{aligned}\frac{1}{E(D)}f(x)&=\int_{-\infty}^{\infty}f(x-t)dt\int_{-\infty}^{\infty}H(y)\ e^{t-y}K(y-t)dy\\ &=\int_{-\infty}^{\infty}H(y)dy\int_{-\infty}^{\infty}f(x-t)\ e^{t-y}K(y-t)dt. \quad (6.2)\end{aligned}$$

To justify this exchange of integrals it will be enough to show that

$$\int_{-\infty}^{\infty}|f(x-t)|dt\int_{-\infty}^{\infty}|H(y)\ e^{t-y}K(y-t)|dy<\infty. \quad (6.3)$$

But $e^{t}K(-t)\ \epsilon\ \{\gamma',\ \delta'\}$, $1-b<\gamma'<\gamma<\delta<\delta'<1-a$, by (6.1). Since $(1-b,\ 1-a)\subset I_F$ we may apply the lemma to show

that the inner integral belongs to $\{\gamma', \delta'\}$. Also $f(x) \varepsilon \{\delta, \gamma\}$ by hypothesis 5, and another application of the lemma shows that the function (6.3) belongs to $\{\delta, \gamma\}$. Thus the inequality (6.3) is established.

Now apply he operator $F(D)$ to both sides of equation (6.2). The left side becomes $\phi(x)$ by Theorem 1; the right side becomes

$$\int_{-\infty}^{\infty} f(t)\, e^{x-t}\, K(t-x)\, dt,$$

and the theorem is proved.

For convenience in application, we restate Theorem 2 in a form which is useful when the range is $(0, \infty)$ rather than $(-\infty, \infty)$.

THEOREM 2*. *Let $M(s)$ satisfy the conditions of Theorem 2 and let $\psi(x)$ and $g(x)$, continuous on $0 \leqslant x < \infty$, satisfy the conditions*

$$\begin{aligned} \psi(x),\ g(x) &= O(x^{\beta-1}) && x \to \infty \\ &= O(x^{\alpha-1}) && x \to 0+,\quad a < \alpha < \beta < b. \end{aligned} \tag{6.4}$$

Then $M(s)$ is the Mellin transform of a function $k(t)$,

$$M(s) = \int_0^{\infty} t^{s-1}\, k(t)\, dt \qquad a < \sigma < b,$$

and if

$$g(x) = \int_0^{\infty} k(xt)\, \psi(t)\, dt \qquad 0 < x < \infty,$$

then

$$\psi(x) = \int_0^{\infty} k(xt)\, g(t)\, dt \qquad 0 < x < \infty.$$

This result follows at once from Theorem 1 if we set $g(e^{-x}) = f(x)$, $k(e^{-x}) = K(x)$ and $e^{x}\,\psi(e^{x}) = \phi(x)$.

As an example, take $k(t) = \sin t$. Then $M(s)$ is the function (1.8). We saw in §3 that we may take $a = -1$ and $b < -\frac{1}{2}$. Hence conditions (6.4) become

$$\begin{aligned} \psi(x),\ g(x) &= O(x^{-\frac{3}{2}-\varepsilon}) && x \to \infty,\ \varepsilon > 0 \\ &= O(x^{-2+\varepsilon}) && x \to 0+. \end{aligned}$$

A pair of Fourier sine transforms satisfying these conditions is

$$g(x) = \sqrt{\frac{2}{\pi}}\frac{2x(3-x^2)}{(x^2+1)^3},\ \psi(x) = x^2e^{-x},$$

obtained by differentiating twice the equation

$$\sqrt{\frac{2}{\pi}}\frac{x}{x^2+1} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin xt\, e^{-t}dt.$$

7. **Applications.** For the Fourier sine transform, let us interpret Theorem 1 by use of the inversion (1.6). We have seen that for this case $M(s)$ is the function (1.8). Different interpretations will result according to the manner in which the functions $E(s)$ and $F(s)$ are selected. Perhaps the most natural choice is

$$E(s) = \sqrt{\frac{2}{\pi}}\sin\frac{\pi s}{2},\quad F(s) = \frac{1}{\Gamma(s)}.$$

Since $a = -1$, $b < -\frac{1}{2}$, we must choose the intervals I_E and I_F accordingly. We take I_E to be $(-2, 0)$ and I_F to be $(-1, 0)$. Since

$$\frac{1}{E(s)} = \frac{2}{\pi}\int_{-\infty}^\infty \frac{e^{-st}}{1+e^{2t}}dt \qquad -2 < \sigma < 0,$$

we have

$$\frac{1}{E(D)}f(x) = \frac{2}{\pi}\int_{-\infty}^\infty \frac{f(x-t)}{1+e^{2t}}dt. \tag{7.1}$$

This integral on the right, after an exponential change of variable, must be essentially a Laplace transform, since an application of $1/\Gamma(D)$ to equation (7.1) gives $\phi(x)$. It is actually an integral of the type

$$\int_0^\infty (e^{-xt}-1)\,\omega(t)dt,$$

but this integral is also inverted by the operator (1.6). We are thus led to the following result.

THEOREM 3. *If $\psi(t)$ is a continuous function such that*

1. $\psi(t) = O(t^{\beta-1}) \qquad t\to\infty$
$\quad = O(t^{\alpha-1}) \qquad t\to 0+,\ -1<\alpha<\beta<-\frac{1}{2}$

and if

2. $$g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xt\, \psi(t) dt,$$

then the following convergent integral defines a function $L(x)$,

$$L(x) = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{g(xt)}{t(1+t^2)} dt$$

and

$$\lim_{k \to \infty} \frac{(-1)^k}{k!} L^{(k)}(k/t)\,(k/t)^{k+1} = \psi(t) \quad 0 < t < \infty.$$

As an example, choose $\psi(t) = e^{-t}$. It satisfies conditions 1. Then

$$g(x) = \sqrt{\frac{2}{\pi}} \frac{x}{x^2+1};\; L(x) = -\frac{2}{\pi} \int_0^\infty \frac{x}{(x^2 t^2+1)(t^2+1)}\, dt.$$

This may be computed by the calculus of residues. Thus

$$L(x) = -\frac{x}{x+1};\; \frac{(-1)^k}{k!} L^{(k)}(k/t)\,(k/t)^{k+1} = \frac{1}{(1+t/k)^{k+1}}.$$

This function clearly tends to e^{-t} when $k \to \infty$ as predicted by the theorem. Another result is obtained by choosing

$$E(s) = \sqrt{\frac{2}{\pi}} \frac{\sin(\pi s/2)}{s},\; F(s) = \frac{1}{\Gamma(s+1)}$$

as follows.

THEOREM 4. *If hypotheses* 1 *and* 2 *of Theorem* 3 *are satisfied then the following convergent integral defines a function* $L(x)$,

$$L(x) = 2\sqrt{\frac{2}{\pi}} \int_0^\infty g(xt) \frac{t}{(t^2+1)^2} dt$$

and

$$\lim_{k \to \infty} \left[\frac{(-x)^{k+1}}{k!} [L(x)/x]^{(k-1)} \right]_{x = k/t} = \psi(t) \quad 0 < t < \infty.$$

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ON A THEOREM OF GELFAND AND HILLE

By V. RAMASWAMI

Introduction and Notation. An ingenious proof has recently been given by Stone[1] of a well-known theorem of Gelfand and Hille, which may be stated as follows:

Let A be a complete, complex or real, Algebra with z and e for its zero and unit elements respectively, with multiplication which is linear and power-associative, and with norm $N(x)$ satisfying

$$N(x) = 0 \text{ if } x = z\,;\ N(x) > 0 \text{ if } x \neq z\,; \tag{1}$$

$$N(sx) = (\text{mod } s)\, N(x) \text{ for every scalar } s\,; \tag{2}$$

$$N(x+y) \leqslant N(x) + N(y)\,;\ N(xy) \leqslant N(x)N(y)\,; \tag{3}$$

and let x be an element of A satisfying

$$\lim_{t\to\infty} [N(x^t)]^{1/t} = 0. \tag{4}$$

Then, a necessary and sufficient condition that $x^{n+1} = z$ for $n \geqslant k \geqslant 0$, (respectively that $x^n = z$ for $n \geqslant k \geqslant 1$) is that

$$\limsup_{p\to\infty} N\left[\sum_{t=1}^{p} (-1)^t (n, t) x^t + e\right] = O(n^k),$$

$$[\text{respectively} \quad = o(n^k)] \tag{5}$$

as mod (n) tends to infinity. Here and in the sequel n denotes any integer, positive or not, and m, t, p, q denote integers $\geqslant 0$ and $(n, t) = n(n-1) \ldots\ldots (n-t+1)/t!$ for $t \geqslant 1$, and $(n, 0) = 1$.

An adaptation of the idea of the basic result of Stone's paper (employed by him for his generalization of Polya's theorem) has led to the following results of this paper:

THEOREM I. *Let B be a complex or real normed linear space (whose elements we denote by x, y with or without suffixes) with z for its zero element, and with norm satisfying*

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$$\textit{the conditions (1) and (2) above,} \tag{6}$$

and

$$\mathcal{N}\left(\sum_{t=1}^{p} y_t\right) \leqslant g\left[\sum_{t=1}^{p} \mathcal{N}(y_t)\right] \text{ for } p \geqslant 1, \tag{7}$$

where $g(\alpha)$ *is a monotonic increasing function of the positive variable* α *and* $g(\alpha) \to 0$ *as* $\alpha \to +0$.

(Actually the condition on g is unnecessary. For, from

$$\mathcal{N}\left(\sum_1^n x_t\right) \leqslant g\left[\sum_1^n \mathcal{N}(x_t)\right] \tag{7a}$$

follows, if at least one $x_t \neq z$, and therefore $S = \sum_1^n \mathcal{N}(x_t) > 0$ that

$$\mathcal{N}\left(\sum_1^n x_t\right) = S \cdot \mathcal{N}\left[\sum_1^n (x_t/S)\right] \leqslant g(1) \sum_1^n \mathcal{N}(x_t) \tag{7b}$$

which is also obviously true if every $x_t = z$, and therefore is equivalent to (7), since obviously $g(1) \geqslant 1$.

It has been pointed out by Stone that if

$$\| x \| = \inf_{x = \sum_1^n x_t} \Sigma \mathcal{N}(x_t), \text{ then } \| x+y \| \leqslant \| x \| + \| y \| ;$$

also by (7*b*), we have $\mathcal{N}(x) \geqslant \| x \| \geqslant \mathcal{N}(x)/g(1)$. It follows that under this *new* norm, the spaces B can be completed, if necessary, without in any way affecting the conditions in the theorems of this paper. This remark is useful in connection with Theorem 2).

Let x_t, $t \geqslant 0$ *be a sequence in* B *satisfying*

$$\lim_{t \to \infty} [\mathcal{N}(x_t)]^{1/t} = 0. \tag{8}$$

Then, a necessary and sufficient condition that $x_{n+1} = z$ *for* $n \geqslant k \geqslant 0$, *(respectively* $x_n = z$ *for* $n \geqslant k \geqslant 1$*) is that*

$$\limsup_{p \to \infty} \mathcal{N}\left[\sum_{t=0}^{p} (-1)^t \, (n,\, t) \; x_t\right] = O(n^k),$$

$$[\textit{respectively } = o\,(n^k)] \tag{9}$$

as mod (n) tends to infinity over any assigned endless arithmetic progression of integers.

We shall denote by $x(n, p)$ the sum in (9) and note that

$$x(m, p) = x(m, m) \text{ for } p \geqslant m \geqslant 0. \tag{9a}$$

THEOREM 2. *Let* x_t, $t \geqslant 0$, *be a sequence in a* B *of the kind considered in Theorem* 1, *and let*

$$\limsup_{t\to\infty} [\mathcal{N}(x_t)]^{1/t} < \infty. \tag{10}$$

Then

(*i*) *a necessary and sufficient condition for* (8) *to hold without every* x_t *being* $= z$ *is that*

$$\limsup_{t\to\infty} \{\mathcal{N}[x(t,t)]\}^{1/t} = 1 = \limsup_{t\to\infty} \{\mathcal{N}[x(t)]\}^{1/t}, \tag{11}$$

where

$$x(t) = \sum_{p=0}^{t} (t, p) x_p;$$

(*ii*) *a necessary and sufficient condition for the conclusion of Theorem* 1 *to hold is that*

$$\mathcal{N}[x(t,t)] + \mathcal{N}[x(t)] = O(t^k), \ [\textit{respectively} = o(t^k)] \ \textit{as } t \to \infty \tag{12}$$

$$\textit{or } \mathcal{N}\left[\sum_{p=0}^{t} (t, 2p+1)\, x_p\right] = O(t^{2k+1}), \ [\textit{respectively} = o(t^{2k+1})] \ \textit{as } t \to \tag{13}$$

$$\textit{or } \mathcal{N}\left[\sum_{p=0}^{t} (t, 2p)\, x_p\right] = O(t^{2k}), \ [\textit{respectively} = o(t^{2k})] \ \textit{as } t \to \infty.$$

It will be noted that the Gelfand-and-Hille Theorem is an obvious consequence of Theorem 1, which, while demanding less of the norm than (3) and (5), is also independent of multiplication, unit element and completeness Theorem 2 while also independent of multiplication and completeness, assumes in some respects more and in some others less of the sequence x_t than Theorem 1, and when applied to any x for which x^t is defined for $t > 0$ provides necessary and sufficient conditions for its nil-potency, generalized or finite.

The proofs of these theorems will be given below for complex B, for, by means of the definitions

$$(a+ib)\,(x+iy) = (ax-by) + i\,(ay+bx) = (x+iy)\,(a+ib)$$

and $\mathcal{N}(x+iy) = \sup_{0\leqslant\theta<2\pi} [\mathcal{N}(x\cos\theta - y\sin\theta) + \mathcal{N}(x\sin\theta + y\cos\theta)]$,

where a and b are any real numbers and x and y are

general elements of a real B satisfying (6) and (7), any such B can be embedded in a complex B satisfying the same relations with $2g(\alpha)$ in place of $g(\alpha)$. Further, the proofs deal only with the sufficiency parts of the theorems, as the necessity parts are obvious

The following further notation is introduced with reference to Theorem 1:

λ: the complex variable

r, δ: positive real numbers between 0 and 1;

$\alpha_r = \cos^{-1}(r/2)$;

$$\phi(r, \lambda) = [1 + r \exp (i\alpha_r)/(\lambda-1)]^{k+1}$$

$$\times [1+r\exp(-i\alpha_r)/(\lambda-1)]^{k+1} = 1 + \sum_{t=1}^{2k+2} \mu(r, t)(\lambda-1)^{-t};$$

$M = \text{Max} \mid \mu(r, t) \mid$, which, obviously, is finite;

$$\beta(\delta) = g[\sum_{t=1}^{\infty} N(x_t)\delta^{-t-1}];$$

$$\beta(m, \delta) = g[\sum_{p+q>m} \{N[(-p-1, q)x_p](1+\delta)^{-q-1}\}];$$

$$\gamma(m, \delta) = g[\sum_{p+q>m} \{N[(-p-1, q)x_p](1-\delta)^{q}\}];$$

$$J(m, \lambda) = \sum_{t=0}^{m} (-1)^t x_t(\lambda-1)^{-t-1};$$

$$b(m, \lambda) = \sum_{t=0}^{m} [x(t, t)\lambda^{-t-1}];$$

$$b(m, p, \lambda) = -\sum_{t=1}^{m} x(-t, p)\lambda^{t-1};$$

u = the positive common difference, and v the first non-negative term in the Arithmetic Progression of (9);

$$\omega_q = \exp \{2\pi iq/u\}; F(m, \lambda) = \sum_{q=0}^{u-1} \omega_q^{v+1} f(m, \lambda\omega_q);$$

$$B(m, \lambda) = \sum_{q=0}^{u-1} \omega_q^{v+1} b(m, \lambda\omega_q);$$

$$B(m, p, \lambda) = \sum_{q=0}^{u-1} \omega_q^{v+1} b(m, p, \lambda\omega_q);$$

$$c(m, \lambda) = \sum_{t=0}^{m} (-1)^t x(t)(\lambda-2)^{-t-1};$$

this symbol will be of use in the proof of Theorem 2.

2. Lemmas subject to (6), (7), (8) and (9).

LEMMA 1. *For*

$|\lambda-1|=r$, $|\phi(r,\lambda)|=O(|\lambda|-1)^{k+1}|\lambda-1|^{-(k+1)}$.

PROOF. Let $\lambda=1+re^{i\theta}$. Then

$$\frac{|\lambda|-1}{|\lambda-1|}=\frac{2|(\cos\theta+\cos\alpha_r)|}{1+|\lambda|}\geqslant\left|\cos\frac{\theta+\alpha_r}{2}.\cos\frac{\theta-\alpha_r}{2}\right|$$
$$=\tfrac{1}{4}\left|1+\frac{re^{i\alpha_r}}{\lambda-1}\right|\left|1+\frac{re^{-i\alpha_r}}{\lambda-1}\right|.$$

LEMMA 2. $\mathcal{N}[f(m,\lambda)]\leqslant\beta(\delta)$, for $|\lambda-1|\geqslant\delta$.
Obvious by (7) and (8).

LEMMA 3. *For fixed* δ, (*i*) $\beta(m,\delta)\to 0$ as $m\to\infty$
(*ii*) $\gamma(m,\delta)\to 0$ as $m\to\infty$.
Obvious by (8) and (7).

LEMMA 4. (*i*) $\mathcal{N}[F(m,\lambda)-B(m,\lambda)]\leqslant u\beta(m,\delta)$ *for* $|\lambda|\geqslant 1+\delta$

(*ii*) $\mathcal{N}[F(m,\lambda)-B(m,p,\lambda)]\leqslant u\gamma(m,\delta)$, *for* $p\geqslant m$, *and* $|\lambda|\leqslant 1-\delta$.

The proof follows from (7), by considering $F(m,\lambda)$ as the sum of the first $m+1$ rows of an absolutely convergent double series, the $(t+1)$th row consisting of the expansion in negative or positive powers of λ of the terms involving x_t in $F(m,\lambda)$.

LEMMA 5. *For* $q\geqslant k$, *and* $m\geqslant 0$, *and* $|\lambda-1|=r$, α_1, α_2 *and* p_m *exist such that*

(*i*) $\mathcal{N}[B(m,\lambda)\,\phi(r,\lambda)\,(\lambda-1)^{q+1}]<\alpha_1$ for $|\lambda|>1$;

(*ii*) $\mathcal{N}[B(m,p_m,\lambda)\,\phi(r,\lambda)\,(\lambda-1)^{q+1}]<\alpha_2$ *for* $|\lambda|<1$ $p_m>m$, *and* α_1 *and* α_2 *are independent of* q *and* r.

PROOF OF (*i*). We observe that $B(m,\lambda)=u\sum_{\substack{0\leqslant t\leqslant m\\ z-t\equiv 0 \bmod u}}x(t,t)\,\lambda^{-t-1}$

and the result follows by (7), (9*a*), (9) and Lemma 1 since $|\lambda|>1$.

PROOF OF (*ii*). We observe that $B(m, p, \lambda)$

$$= -u \sum_{\substack{1 \leqslant t \leqslant m \\ v+t \equiv 0 \bmod u}} x(-t, p)\, \lambda^{t-1}.$$

By (9), p_m exists such that $N[x(-t, p)] = O(t^k)$ for $1 \leqslant t \leqslant m$, $t+v \equiv 0 \bmod u$, and $p \geqslant p_m > m$, and the result follows as in (*i*), since $|\lambda| < 1$.

Before proceeding to the next lemma we have to make a few preliminary observations on functions $\psi(\lambda)$ of the type $\sum_{m=1}^{t} y_m f_m(\lambda)$, where $y_m \epsilon B$ and the $f_m(\lambda)$ are holomorphic functions of λ in any specified region of the complex plane. It is obvious that the notions of contour integration are applicable to such functions and that Cauchy's theorem and Taylor's theorem are valid for such analytic functions. In particular we have

$$\int_c \psi(\lambda)\, d\lambda = \sum_{m=1}^{t} y_m \int_c f_m(\lambda) d\lambda\, ; \int_{c_1} \psi(\lambda) d\lambda + \int_{c_2} \psi(\lambda) d\lambda$$

$$= \int_{c_1+c_2} \psi(\lambda) d\lambda\, ;$$

$$\int_c [\psi_1(\lambda) + \psi_2(\lambda)] d\lambda = \int_c \psi_1(\lambda) d\lambda + \int_c \psi_2(\lambda) d\lambda,$$

and

$N\left[\int_c \psi(\lambda) d\lambda\right] \leqslant g \left\{\int_c N[\psi(\lambda)]\, |d\lambda| + \epsilon\right\}$ for every positive number ϵ.

LEMMA 6. *To $q \geqslant k$, corresponds an m, such that*

$$N\left[\frac{1}{2\pi i} \int_{|\lambda-1|=r} F(m, \lambda)\, \phi(r, \lambda)\, (\lambda-1)^{q+1} d\lambda\right] \leqslant g(r\, \alpha_3),\ m > q,$$

$$0 < r \leqslant \sin\frac{\pi}{4u} \text{ and } \alpha_3 \text{ is independent of } q \text{ and } r.$$

PROOF. We divide the path of integration into four parts with respect to any $\delta < r$; Γ_1 for which $|\lambda| \geqslant 1+\delta$; Γ_2 for which $|\lambda| \leqslant 1-\delta$; Γ_3 and Γ_4 for which $1-\delta < |\lambda| < 1+\delta$. We denote by I_1, I_2, I_3, I_4 respectively the integrals over these parts. Let $l(\delta)$ be the length of each of Γ_3 and Γ_4. It is obvious that $l(\delta) \to 0$ as $\delta \to 0$; we may, therefore

suppose δ so chosen that $u\,\beta(r)2^{2k+2}r^q l(\delta) < 1$, where $\beta(r)$ is the number provided by Lemma 2. Then, by (15)

$$\mathcal{N}(I_3) \leqslant r\,g(1);\ \ \mathcal{N}(I_4) \leqslant r\,g(1). \tag{16}$$

Now

$$I_1 = r\int_{\Gamma_1}\{[F(m,\lambda)-B(m,\lambda)]+B(m,\lambda)\}\phi(r,\lambda)\frac{(\lambda-1)^{q+1}}{r}d\lambda.$$

Hence, by (15) and Lemmas 4 and 5 follows

$$\mathcal{N}(I_1) \leqslant rg\left\{\int_{\Gamma_1}[u\,\beta(m,\delta)+\alpha_1]\frac{|d\lambda|}{r}+1\right\} \leqslant rg[u\,\beta(m,\delta)+\alpha_1+1]. \tag{17}$$

Also

$$I_2 = r\int_{\Gamma_2}\{[F(m,\lambda)-B(m,p_m,\lambda)]+B(m,p_m,\lambda)\}$$
$$\times\,\phi(r,\lambda)\frac{(\lambda-1)^{q+1}}{r}d\lambda,$$

where the p_m is taken from Lemma 5.

Hence, by (15) and Lemmas 4 and 5 again

$$\mathcal{N}(I_2) \leqslant rg\left\{\int_{\Gamma_2}[u\gamma(m,\delta)+\alpha_2]\frac{|d\lambda|}{r}+1\right\} \leqslant rg[u\gamma(m,\delta)+\alpha_2+1]. \tag{18}$$

Now by Lemma 3, m obviously exists such that

$$m > q;\ \ \beta(m,\delta) < 1;\ \ \gamma(m,\delta) < 1. \tag{19}$$

The lemma now follows from (15), (16), (17), (18) and (19).

LEMMA 7. *The function $F(m,\lambda)$ is analytic except for poles at the u th roots of unity, and elements $y_{m,n}$ of B and positive constants $d(n)$ exist such that*

$$F(m,\lambda) = \sum_{n\geqslant -m} y_{m,n}(\lambda-1)^{n-1}\ \textit{for}\ \alpha|\lambda-1| = r \leqslant \sin\pi/4u; \tag{20}$$

$$y_{m,-t} = (-1)^t x_t\ \text{for}\ m \geqslant t \geqslant 1, \tag{21}$$

$$\mathcal{N}(y_{m,n}) < d(n). \tag{22}$$

PROOF. The lemma is obvious in case $u=1$. In case $u>1$, we observe that each of the functions $f(m,\chi\omega_q)$,

$1 \leqslant q \leqslant u-1$ is holomorphic inside and on $|\lambda-1| = \sin \pi/4u$, and that by Lemma 2 the norm of each in this closed region is $\leqslant \beta(\sin \pi/4u)$. The desired result then follows from Taylor's theorem.

3. Proof of Theorem 1.

O-Case. Let $q \geqslant k$.

For the m provided by Lemma 6 we have, by the theory of residues

$$(-1)^{q+1}x_{q+1} = \frac{1}{2\pi i}\int\limits_{|\lambda-1|=r \leqslant \sin \pi/4u} F(m, \lambda)\phi(r, \lambda)(\lambda-1)^{q+1}d\lambda$$

$$-\sum_{t=1}^{2k+2} \mu(r, t)\, y_{m,\, t-q-1} r^t.$$

Hence, by (7) and Lemmas 6 and 7

$$\mathcal{N}(x_{q+1}) \leqslant g\left\{ g(r\alpha_3) + M\sum_{t=1}^{2k+2} d(t-q-1)r^t \right\} \to 0 \text{ as } r \to 0, \text{ by (7)}.$$

Hence, $\qquad x_{q+1} = z.$

o-case. We have, by what has been proved for the O-case, $x_q = z$ for $q \geqslant k+1$ and hence $(-1)^k x_k = \lim\limits_{m\to\infty} m^{-k}x(m, m) = z$, by the o-condition of (9).

4. Proof of Theorem 2.

Proof of (*i*). We regard B as complete in virtue of the remarks in (7), and write

$$b(\lambda) = \lim_{m\to\infty} b(m, \lambda) \quad \text{for } |\lambda| > 1$$

$$c(\lambda) = \lim_{m\to\infty} c(m, \lambda) \quad \text{for } |\lambda-2| > 1$$

$$f(\lambda) = \lim_{m\to\infty} f(m, \lambda) \quad \text{for } |\lambda| > \limsup_{t\to\infty} [\mathcal{N}(x_t)]^{1/t}.$$

These limits exist on account of completeness of B and (10) and (11). For sufficiently large $|\lambda|$ it is easily seen by the double series method that these functions are equal. By the theory of analytic continuation it follows that $b(\lambda)-c(\lambda) = z$ in the region $|\lambda| > 1$, $|\lambda-2| > 1$. Hence, there exists an analytic function $\psi(\lambda)$ defined

everywhere except at $\lambda = 1$ and equal to $f(\lambda)$ for sufficiently large $|\lambda|$. If $M(r)$ is an upper bound for the norm of this function on $|\lambda - 1| = r$ we have, by Cauchy's theorem

$$(-1)^t x_t = \frac{r^t}{2\pi i} \int_{|\lambda - 1| = r} \psi(\lambda) \frac{(\lambda-1)^t}{r^t} d\lambda \, ;$$

$$N(x_t) \leqslant r^t g\left[\frac{1}{2\pi}\int_0^{2\pi} r\, M(r) d\theta + 1\right] = r^t g[rM(r)+1];$$

and hence

$$\limsup_{t\to\infty} [N(x_t)]^{1/t} \leqslant r.$$

This being true for every $r > 0$, (8) follows.

By integrations on $|\lambda| = \rho > 1$ and $|\lambda - 2| = \rho > 1$, it follows as above that

$$\limsup_{t\to\infty} [N\{x(t; t)\}]^1/^t \leqslant 1, \quad \limsup_{t\to\infty} [N\{x(t)\}]^{1/t} \leqslant 1.$$

If the inequality held in either of these cases, it would follow that 1 is a regular point for one of $b(\lambda)$ and $c(\lambda)$, and by the theory of analytic continuation also for $f(\lambda)$, and hence by Taylor's theorem that every $x_t = z$.

Proof of (*ii*). *O-Case.* Here, obviously

$$\limsup_{t\to\infty} [N\{x(t, t)\}]^{1/t} \leqslant 1, \quad \limsup_{t\to\infty} [N\{x(t)\}]^{1/t} \leqslant 1.$$

Hence the functions $b(\lambda)$ and $c(\lambda)$ of (*i*) are defined for $|\lambda| > 1$ and $|\lambda - 2| > 1$ respectively and as above are equal to $f(\lambda)$ for sufficiently large $|\lambda|$. Hence as above

$$\limsup_{t\to\infty} [N(x_t)]^{1/t} = 0.$$

From this point the proof proceeds mutatis mutandis as in Theorem 1, with $c(m, \lambda)$ in place of $b(m, \lambda)$, and with

$$\psi(r, \lambda) = \left(\frac{1 - r^2 e^{2i\alpha_r}}{(\lambda-1)^2}\right)^{k+1} \left(\frac{1 - r^2 e^{-2i\alpha r}}{(\lambda-1)^2}\right)^{k+1}$$ instead of $\phi(r, \lambda)$.

The path of integration $|\lambda - 1| = r$ is divided into the parts Γ_1 and Γ_2 for which the real part of $\lambda - 1$ is non-negative and negative respectively: and on Γ_1 we have

$$N[b(m, \lambda)\, \psi(r, \lambda)\, (\lambda-1)^{q+1}] < \alpha_4 \text{ for } q \geqslant k\, ;$$

$$N[f(m, \lambda) - b(m, \lambda)] < \beta\, [m, (\sqrt{1+r^2} - 1)]\, ;$$

and on Γ_2, we have

$$\mathcal{N}[c(m, \lambda)\, \psi(r, \lambda)\, (\lambda-1)^{q+1}] < \alpha_5 \text{ for } q \geqslant k;$$
$$\mathcal{N}[f(m, \lambda)-c(m, \lambda)] < \beta\,[m, (\sqrt{1+r^2}-1)],$$

where α_4, α_5 are independent of q and r and for fixed δ, $\beta(m, \delta) \to 0$ as $m \to \infty$.

This proves the sufficiency in the O-case of (12) and the sufficiency in the o-case follows as in Theorem 1. The necessity in both cases is obvious.

To prove (13) we consider the sequence y_t defined for $t \geqslant 0$ by the relations $y_m = z$ if m is even and $y_{2m+1} = x_m$ for $m \geqslant 0$ and apply (12) noting that $y(t, t)+y(t) = z$.

To prove (14) we consider the sequence y_t defined for $t \geqslant 0$ by the relations $y_{2m} = x_m$ for $m \geqslant 0$, $y_{2m+1} = z$ for for $m \geqslant 0$; and again apply (12) noting that in this case $y(t, t) = y(t)$.

In conclusion I wish to record my thanks to Prof. Stone for pointing out, as indicated in (7), that Theorem 2 is independent of completeness.

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SOME THEOREMS FOR INFINITE SYSTEMS OF LINEAR EQUATIONS

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We deal initially with a homogeneous triangular system of linear equations in infinitely many unknowns $\beta_0, \beta_1, \ldots$. We may suppose that the leading coefficient is unity so that the system can be written in the form

$$\beta_p + \sum_{k=p+1}^{\infty} a_{p,k}\beta_k = 0 \ (p = 0, 1, 2, \ldots). \tag{1}$$

Perron [4] has discussed this system and its non-homogeneous case under the hypothesis $\Sigma |a_{pk}| < \infty$, while von Koch [5] and Walsh [6] have discussed them under the respective hypotheses

$$|a_{pk}| \leqslant \delta^{k-p} \text{ and } \sum_{k=p+1}^{\infty} |a_{p,k}|^q \leqslant P^q (p = 0, 1, 2..).$$

In the present paper, the system (1) and the corresponding non-homogeneous system will be discussed under less stringent conditions on the size of the coefficients, it being supposed only that they satisfy an inequality of the form $|a_{pk}| \leqslant \psi(p)\phi(k)$. By this means we shall arrive at extensions of the results of the aforementioned writers, and at theorems which are valid for all systems (1). In the final part, the theorems of the first sections are applied to a class of interpolation problems for analytic functions. As a general result, it is shown that every interpolation problem generated by a triangular set of linear operators possesses a solution provided that the interpolated values are sufficiently small. Moreover, this solution is unique in a certain sense.

1. **The Homogeneous case.** In our study of the system (1) it will be of fundamental importance to introduce the following determinants formed from the coefficients a_{pk}. For integers p, n, m, $p \geqslant 0$, $n \geqslant 0$, $m \geqslant p+n-1$,

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we shall designate by $\gamma^p_{n\,m}$ the following n by n determinant

$$\gamma^p_{n\,m} = \begin{vmatrix} a_{p\,p+1} & a_{p\,p+2} & \cdots & a_{p\,p+n-1} & a_{p\,m} \\ a_{p+1\,p+1} & a_{p+1\,p+2} & \cdots & a_{p+1\,p+n-1} & a_{p+1\,m} \\ \cdot & \cdot & & \cdot & \cdot \\ a_{p+n-1\,p+1} & a_{p+n-1\,p+2} & \cdots & a_{p+n-1\,p+n-1} & a_{p+n-1\,m} \end{vmatrix} \quad (n \geqslant 2) \tag{2}$$

$$\gamma^p_{1\,m} = a_{pm}, \; \gamma^p_{0m} = 1, \tag{2'}$$

In (2) we have extended the definition of the coefficients a_{ph} as follows

$$a_{pk} = 1 \text{ for } p = k, \; a_{pk} = 0 \text{ for } p > k. \tag{3}$$

By way of elucidating the composition of the determinants γ^p_{nm}, we have, as examples,

$$\gamma^1_{24} = \begin{vmatrix} a_{12} & a_{14} \\ 1 & a_{24} \end{vmatrix}, \; \gamma^0_{34} = \begin{vmatrix} a_{01} & a_{02} & a_{04} \\ 1 & a_{12} & a_{14} \\ 0 & 1 & a_{24} \end{vmatrix}. \tag{4}$$

It will be noted from (2) and (3) that in all the determinants $\gamma^p_{n,m}$, the diagonal immediately below the leading diagonal is composed entirely of 1's while the elements to the left of it are all zero. These determinants satisfy a number of identities of which we shall require but two:

$$\gamma^p_{nm} = a_{p+n-1\,m}\,\gamma^p_{n-1\,p+n-1} - \gamma^p_{n-1\,m} \tag{5}$$

which is obtained by expanding the determinant (2) in terms of the elements of its last row, and

$$\gamma^p_{j-p\,j} = \sum_{k=1}^{j-p} (-1)^{k-1} a_{p\,p+k}\,\gamma^{p+k}_{j-p-k\,j}. \tag{6}$$

This may be established by expanding the determinant $\gamma^p_{j-p\,j}$ in terms of the elements of its first row. The importance of these determinants is due to the following lemma.

LEMMA 1. *The quantities β_i satisfy the system* (1) *if and only if for each integer $p \geqslant 0$ and each integer $k \geqslant 1$, the following systems are also satisfied:*

$$(-1)^k \beta_p = \sum_{j=k}^{\infty} \gamma^p_{k\;p+j}\,\beta_{p+j}. \tag{7}$$

PROOF. From the pth equation of (1) eliminate successively the quantities $\beta_{p+1}, \beta_{p+2}, \ldots, \beta_{p+k}$ and obtain the kth equation of (7). Conversely, (7) with $k = 1$ is identical with the pth equation of (1).

Let $\psi(n) \geqslant 0$ and $\phi(n) \geqslant 0$. We shall suppose now that the coefficients a_{pk} of (1) are subject to an inequality of the form

$$|a_{pk}| \leqslant \psi(p)\phi(k) \qquad (k > p). \tag{8}$$

Our next lemma establishes bounds for the size of the determinants $\gamma^p_{n\,m}$.

LEMMA 2. *Under the restriction* (8), *the quantities* γ^p_{nm} *satisfy the following inequality*

$$|\gamma^p_{nm}| \leqslant \psi(p)\,\phi\,(m) \prod_{j=1}^{n-1} (1+\psi\,(p+j)\,\phi\,(p+j)). \tag{9}$$

The indicated product must be set equal to 1 for the case $n = 1$.

PROOF. We use induction on n, the order of the determinant γ^p_{nm}. The inequality is readily established for $n = 1, 2$. Assume that it is true for $n^* = 1, 2, \ldots, n-1$. From (5) there is obtained

$$\begin{aligned}|\gamma^p_{nm}| &\leqslant |a_{p+n-1\,m}|\,|\gamma^p_{n-1\;p+n-1}| + |\gamma^p_{n-1\,m}| \\ &\leqslant \psi(p+n-1)\,\phi(m)\,\psi(p)\phi(p+n-1) \\ &\qquad\qquad \times \prod_{j=1}^{n-2} (1+\psi(p+j)\phi(p+j)) \\ &\quad +\psi(p)\phi(m) \prod_{j=1}^{n-2}(1+\psi(p+j)\phi(p+j)) \\ &= \psi(p)\phi(m) \prod_{j=1}^{n-1} (1+\psi\,(p+j)\,\phi\,(p+j))\end{aligned}$$

which establishes the inequality (9) for $n^* = n$.

LEMMA 2 is in effect a generalization of an inequality given in Whittaker [7] for determinants with the particular 0 and 1 structure of γ^p_{nm}. It should be remarked that the right hand side of the inequality may actually be attained. This will be the case, for instance, when $a_{pk} = (-)^{p+k+1}\,\psi(p)\phi(k)$.

THEOREM 1. *Let* $H(n) = \prod_{j=1}^{n-1} (1+\psi(j)\,\phi(j))$, *and let* $R(n) = R(n;\,\beta) = \sum_{j=n}^{\infty} |\beta_j|\,\phi(j)$. *If the system* (1) *is subject to the inequality* (8), *then the only solution of* (1) *for which*

$$\liminf_{n\to\infty} H(n)\,R(n) = 0 \tag{10}$$

is the trivial solution $\beta_i \equiv 0$.

PROOF. Let the integer $p \geqslant 0$ be fixed. By Lemma 1 we have

$$|\beta_p| \leqslant \sum_{j=k}^{\infty} |\gamma^p_{k\,p+j}|\;|\beta_{p+j}| \qquad (k = 1,\,2,\,\ldots).$$

Applying Lemma 2, we get

$$|\beta_p| \leqslant \sum_{j=k}^{\infty} |\beta_{p+j}|\,\psi(p)\,\phi(p+j) \prod_{t=1}^{k-1} (1+\psi(p+t)\,\phi(p+t))$$
$$= \psi(p)[H(p+1)]^{-1} \sum_{j=k}^{\infty} |\beta_{p+j}|\,\phi(p+j)\,H(p+k)$$
$$= \psi(p)[H(p+1)]^{-1}\,R(p+k)\,H(p+k).$$

Now by (10) we have $\lim_{i\to\infty} H(n_i)R(n_i) = 0$ for some sequence of integers n_i. If now we let k take on the values k_i for which $p+k_i = n_i$, we see that β_p is dominated by a null sequence and is consequently zero.

COROLLARY 1.1. *Given a system* (1), *there exist positive constants* $\chi_n > 0$ *such that the only solution of* (1) *subject to* $|\beta_n| \leqslant \chi_n$ $(n = 0,\,1,\,2,\,\ldots)$ *is the trivial solution.*

PROOF. Let

$$M_k = \text{Max}\,(|a_{0k}|,\,|a_{1k}|,\,\ldots,\,|a_{k-1\,k}|) \quad (k = 1,\,2,\ldots) \tag{11}$$

then (8) will be satisfied with $\psi(p) = 1$, $\phi(k) = M_k$. By our theorem, the only solution of (1) subject to

$$\liminf_{n\to\infty} \prod_{j=1}^{n-1} (1+M_j) \sum_{k=n}^{\infty} M_k|\beta_k| = 0 \tag{12}$$

is the trivial solution. Now let Σd_n be an arbitrary convergent series of positive terms. Then it is sufficient to take

$$x_n = d_n \left[\prod_{j=1}^{n} (1+M_j) \right]^{-1}.$$

For in this case we shall have

$$\sum_{k=n}^{\infty} M_k |\beta_k| \leqslant \sum_{k=n}^{\infty} d_k M_k \left[\prod_{j=1}^{k} (1+M_j) \right]^{-1}$$
$$\leqslant \left[\prod_{j=1}^{n-1} (1+M_j) \right]^{-1} \sum_{k=n}^{\infty} d_k,$$

so that the left side of (12) is dominated by $\sum_{k=n}^{\infty} d_k$ which is $o(1)$ as $n \to \infty$.

There may exist non-trivial solutions of (1) for which the limit (12) is not 0. For instance, let $M_j \geqslant \epsilon > 0$ $(j = 1, 2, \ldots)$. If we make the special choice of coefficients $a_{ij} = (-1)^{i+j+1} M_j$, then it may be easily verified that $\beta_0 = -1, \beta_n = (-1)^{n+1}/(1+M_1)(1+M_2)\ldots(1+M_n)$ $(n \geqslant 1)$ is a solution of (1) for which the limit (12) equals 1.

From the very proof of Corollary 1.1 there are seen to be infinitely many sets $\{\beta_n\}$. We shall call such a set *a positive restriction*, and in this terminology then, every triangular system (1) possesses *a positive restriction.* In a later section we shall investigate the converse question of identifying those systems which possess positive (and weaker) restrictions.

COROLLARY 1.2. *Let M_k be defined as in (11). Suppose further that*

$$\sum_{k=1}^{\infty} M_k < \infty; \tag{13}$$

then the only solution of (1) for which

$$\sum_{k=1}^{\infty} M_k |\beta_k| < \infty \tag{14}$$

is the trivial solution. Hence, in particular, the only bounded solution is the trivial solution.

PROOF. In virtue of (13), $\prod_{k=1}^{\infty} (1+M_k)$ converges and therefore its partial products are bounded. Condition (14) now insures that the limit of (12) is zero.

Corollary 1.2 generalizes a result of Perron wherein the convergence of $\sum_{p,k} |a_{pk}|$ is required, inasmuch as the convergence of the latter implies (13), but not conversely. There may exist non-trivial solutions for which (14) fails. By way of example, consider the system $\beta_p-(p+1)^{-2}\beta_{p+1}=0$ ($p=0, 1, 2, \ldots$). This possesses the non-trivial solution $\beta_p=(p!)^2$ for which the series (14) is divergent.

COROLLARY 1.3. *With M_k defined as in* (11), *suppose that*

$$\limsup_{n\to\infty} [(1+M_1)(1+M_2)\ldots(1+M_n)]^{1/n}=\sigma<\infty, \quad (15)$$

then the only solution of (1) *for which*

$$\limsup_{n\to\infty} |\beta_n|^{1/n}<\sigma^{-1} \quad (16)$$

is the trivial solution.

PROOF. We have $H(n)\dot{R}(n) \leqslant \sum_{k=n}^{\infty} |\beta_k| H(k+1)$. By (16) there is an ϵ with $0<\epsilon<\sigma^{-1}$ and such that $|\beta_n| \leqslant (\sigma^{-1}-\epsilon)^n$ for all $n \geqslant n_0$. Select $0<\epsilon'<\sigma^2\epsilon/(1-\sigma\epsilon)$. Then by (15) there is an n_1 such that $H(n+1) \leqslant (\sigma+\epsilon')^n$ for $n \geqslant n_1$. Hence for $n \geqslant \max(n_0, n_1)$, we have

$$|\beta_k| H(k+1) \leqslant [(\sigma^{-1}-\epsilon)(\sigma+\epsilon')]^k \leqslant \theta^k, \ \theta<1.$$

Therefore

$$\sum_{k=n}^{\infty} |\beta_k| H(k+1)=o(1), \ n\to\infty.$$

It is to be remarked that the strict inequality in (16) cannot be weakened. For if, in the example mentioned under Corollary 1.1, we take $M_j \equiv M>0$, then $\sigma=1+M$, while $\beta_n=(-1)^{n+1}/(1+M)^n$ is a non-trivial solution for which $\lim_{n\to\infty} |\beta_n|^{1/n}=\sigma^{-1}$. Similarly, by considering rows instead of columns we arrive at the following

COROLLARY 1.4. *Let*

$$N_k=\operatorname{lub}_{j>k} |a_{kj}| \qquad (k=0, 1, 2,\ldots) \quad (17)$$

and suppose that

$$\limsup_{n\to\infty} [(1+N_0)(1+N_1)\dots(1+N_{n-1})]^{1/n} = \sigma < \infty. \quad (18)$$

Then the only solution of (1) *for which* (16) *holds is the trivial solution. The strict inequality in* (16) *cannot be weakened.*

COROLLARY 1.5. *Let* $\alpha > 0$, $\delta > 0$ *and* (*a*) $\alpha\delta < 1$, *or* (*b*) $\alpha\delta = 1$. *Suppose furthermore that the coefficients* a_{pk} *of* (1) *satisfy the inequality*

$$|a_{pk}| \leqslant \alpha^p \delta^k \qquad (k > p). \quad (19)$$

Then the trivial solution is the only solution of (1) *for which*

$$\limsup_{n\to\infty} |\beta_n|^{1/n} < 1/\delta; \quad (20a)$$

or

$$\limsup_{n\to\infty} |\beta_n|^{1/n} < 1/2\delta \quad (20b)$$

PROOF. (*a*) With $\psi(n) = \alpha^n$, $\phi(m) = \delta^m$, we have $H(n)$ bounded in virtue of condition (*a*). By (20*a*) the series $\Sigma |\beta_k| z^k$ converges for $z = \delta$ so that (10) holds. (*b*) By (12) the only solution for which $\liminf_{n\to\infty} 2^n \sum_{k=n}^{\infty} |\beta_k| \delta^k = 0$, hence, in particular, for which (20*b*) holds is the trivial solution. This latter result is implicit in the work of v. Koch [5].

2. The Non-Homogeneous Case. We turn our attention next to non-homogeneous triangular systems

$$\beta_p + \sum_{k=p+1}^{\infty} a_{pk}\beta_k = c_p \quad (p = 0, 1, 2, \dots). \quad (21)$$

If by means of the 1st, 2nd, ..., and qth equations of (21) we eliminate successively the quantities β_1 $\beta_2, \dots, \beta_q$ then we shall obtain an equation of the form β_0+linear combination of $\beta_{q+1}, \beta_{q+2}, \dots$ = linear combination of $c_0, c_1, \dots, c_q$. If now, we allow $q \to \infty$, then formally there will be obtained β_0 = linear combination of $c_0, c_1, \dots$ Similar results hold for β_r. We now proceed to make this device rigorous. We shall show that under certain conditions,

$$\beta_v = \sum_{j=v}^{\infty} (-1)^{j-v} \gamma_{j-v}^{v} c_j \quad (v = 0, 1 \dots) \quad (22)$$

is a solution of (21). Now this is true formally at least. For if (22) is substituted into the pth equation of (21), there is obtained

$$\beta_p + \sum_{k=1}^{\infty} a_{p,\,p+k} \sum_{j=p+k}^{\infty} \gamma_{j-p-k,\,j}^{p+k} \;(-1)^{j-p-k}\, c_j$$

$$= \beta_p - \sum_{j=p+1}^{\infty} (-1)^{j-p} c_j \sum_{k=1}^{j-p} a_{p,\,p+k}\, \gamma_{j-p-k,\,j}^{p+k} \;(-1)^{k-1}$$

$$= \beta_p - \sum_{j=p+1}^{\infty} (-1)^{j-p} \gamma_{j-p,\,j}^{p}\; c_j \qquad \text{by } (6)$$

$$= c_p \text{ by } (22).$$

The interchange of summation signs is valid provided we can show that

$$\sum_{j=p+1}^{\infty} |c_j| \sum_{k=1}^{j-p} |a_{p,\,p+k}|\, |\gamma_{j-p-k,\,j}^{p+k}| < \infty \quad (p = 0, 1, \ldots). \tag{23}$$

Our theorems will be based on an estimate which we shall give for the inner series of (23). In virtue of (2′) we may write the inner series as follows

$$|a_{pj}| + |a_{p,\,j-1}|\,|a_{j-1,\,j}| + \sum_{k=1}^{j-p-2} |a_{p,\,p+k}|\,|\gamma_{j-\,-}^{p+k}\,{}_{,\,j}|. \tag{24}$$

If we now assume that the inequality (8) holds, then it follows from Lemma 2 that (24) is less than the following expression

$$\psi(p)\phi(j) + \psi(p)\phi(j-1)\psi(j-1)\phi(j) + \sum_{k=1}^{j-p-2} \psi(p)\phi(p+k)\psi(p+k)\phi(j) \prod_{t=1}^{j-p-k-1} (1+\psi(p+k+t)\,\phi(p+k+t)) \tag{25}$$

$$= \psi(p)\phi(j)\Big\{1 + \psi(j-1)\phi(j-1) + \sum_{k=1}^{j-p-2} \psi(p+k)\phi(p+k) \prod_{t=1}^{j-p-k-1} (1+\psi(p+k+t)\phi(p+k+t))\Big\}$$

$$= \psi(p)\phi(j)\, H(j)\, [H(p+1)]^{-1}$$

as a forma' computation shows.

THEOREM 2. *Let the coefficients of the system* (21) *be subject to the inequality* (8). *Then if*

$$\sum_{j=1}^{\infty} |c_j|\, \phi(j)\, H(j) < \infty \tag{26}$$

the quantities β_j *given by* (22) *constitute a solution.*

PROOF. By (22) and (9),

$$|\beta_v| \leqslant \psi(v)[H(v+1)]^{-1} \sum_{j=v}^{\infty} |c_j| \phi(j) H(j) \qquad (v = 0, 1, ..), \tag{27}$$

so that by (26), the series (22) all converge absolutely. Furthermore, by (25), the series (23) is less than

$$\psi(p)\ [H(p+1)]^{-1} \sum_{j=p+1}^{\infty} |c_j| \phi(j)\ H(j)$$

and so converges absolutely for all p.

Corresponding to the conditions given in each corollary of Theorem 1, we may obtain an analogous corollary for Theorem 2. We shall state only two explicitly.

COROLLARY 2.1. *Given a system* (21) *in which the* a_{pk} *are considered fixed. Then there exist two sets of positive constants* $\chi_n^{(1)} > 0$, $\chi_n^{(2)} > 0$ $(n = 0, 1, 2, \ldots)$ *such that if* $|c_n| \leqslant \chi_n^{(1)}$, *the system possesses a solution given by* (22), *and this is the sole solution for which* $|\beta_n| \leqslant \chi_n^{(2)}$.

PROOF. With M_k defined as in (11), set $\psi(p) = 1$, $\phi(k) = M_k$. Then $H(j) = \prod_{k=1}^{j-1} (1+M_k)$. Let $d_j^{(2)}$ be a positive and strictly decreasing sequence for which $\sum_{j=0}^{\infty} d_j^{(2)} < \infty$. Set $d_j^{(1)} = d_j^{(2)} - d_{j+1}^{(2)}$. The $d_j^{(1)}$ are all positive, and furthermore, $\sum_{j=v}^{\infty} d_j^{(1)} = d_v^{(2)}$. It suffices to set $\chi_n^{(1)} = d_n^{(1)}/H(n+1)$, $\chi_n^{(2)} = d_n^{(2)}/H(n+1)$. For with $|c_j| \leqslant \chi_j^{(1)}$,

$$\sum_{j=1}^{\infty} |c_j| \phi(j) H(j) \leqslant \sum_{j=1}^{\infty} d_j^{(1)} < \infty$$

so that by our theorem, (22) is a solution. Now by (27),

$$|\beta_v| \leqslant [H(v+1)]^{-1} \sum_{j=v}^{\infty} d_j^{(1)} = d_v^{(2)}/H(v+1) = \chi_n^{(2)},$$

so that (22) is a solution with $|\beta_v| \leqslant \chi_v^{(2)}$. Now if there were two solutions $\beta_n^{(1)}$ and $\beta_n^{(2)}$ each in absolute value less than $\chi_n^{(2)}$ $(n = 0, 1, 2, \ldots)$, then $\beta_n' = \frac{1}{2}(\beta_n^{(1)} - \beta_n^{(2)})$ would be a solution of the homogeneous system (1), and with

$|\beta^*_n| \leqslant \chi^{(2)}_n$. Referring to the proof of Corollary 1.1, we see that by our choice of $\chi^{(2)}_n$ this implies $\beta^*_n \equiv 0$.

COROLLARY 2.2. *Let M_k be defined as in* (11). *Suppose that*

$$\sum_{k=1}^{\infty} M_k < \infty. \tag{13}$$

If now the constants c_j satisfy

$$\sum_{k=1}^{\infty} M_k |c_k| < \infty \tag{28}$$

then (22) *is a solution of* (21), *and it is the sole solution for which*

$$\sum_{k=0}^{\infty} M_k |\beta_k| < \infty. \tag{14}$$

PROOF. Let $\psi(p) = 1$, $\phi(k) = M_k$. In view of (13), $H(j)$ is bounded so that (28) implies (26). By (28) and (27) β_k is bounded so that (14) holds. Uniqueness follows from Corollary 1.2. This is a generalization of a theorem of Perron for the non-homogeneous case.

3. **On the Existence of Restrictions.** We have shown in Corollary 1.1 that corresponding to every homogeneous triangular system (1) there exists a set of constants $\chi_n > 0$ such that a solution of (1) with $|\beta_n| \leqslant \chi_n$ is necessarily the trivial solution. It is of considerable interest to obtain a characterization of the full linear homogenous systems

$$\sum_{k=0}^{\infty} a_{pk}\beta_k = 0 \qquad (p = 0, 1, 2, \ldots) \tag{29}$$

for which such sets of constants exist, or as we shall say, which possess positive restrictions. We shall find that systems (29) which possess positive restrictions are, essentially, systems which can be reduced to triangular form. In addition, we shall consider semi-positive and non-zero restrictions, i.e. $\chi_n > 0$ for infinitely many n, or χ_n not identically zero, and shall obtain characterizations of

those systems which possess restrictions of each of these weaker types.

By c_{jk} $(j, k = 0, 1, 2, \ldots)$ we shall mean the determinant

$$c_{jk} = \begin{vmatrix} a_{00}, & a_{01}, & \cdots, & a_{0\,j-1}, & a_{0k} \\ a_{10}, & a_{11}, & \cdots, & a_{1\,j-1}, & a_{1k} \\ \cdot & \cdot & & \cdot & \cdot \\ a_{j0}, & a_{j1}, & \cdots, & a_{j\,j-1}, & a_{jk} \end{vmatrix}. \qquad (30)$$

By the principal diagonal minors of the matrix $A = (a_{pk})$, we shall mean the determinants c_{jj} $(j = 0, 1, 2, \ldots)$. In addition, let us now distinguish the following three classes of infinite matrices A.

1. An infinite matrix A will be said to be of class R_1 if it is possible to delete a certain number of rows (0 or ∞ allowed) and rearrange the remaining rows in such a way that the rearranged residual matrix has infinitely many rows and has non-vanishing principal diagonal minors.

2. An infinite matrix A will be said to be of class R_2 if it is possible to delete from A a certain number of columns (0 or ∞ allowed) such that an $\infty \times \infty$ matrix remains and is of class R_1.

3. An infinite matrix is of class R_3, if and only if, it possesses at least one non-zero element, i.e. $A \neq 0$.

We shall now show that these three classes of matrices completely characterize the systems (29) which possesses positive, semi-positive, and non-zero restrictions respectively. We require the following lemma which seems to have been stated explicitly for the first time by Kotteritzsch [3].

Lemma 3. *If none of the principal diagonal minors of the matrix $A = (a_{pk})$ of the systems (29) vanishes, then this system is equivalent to the triangular system*

$$\beta_p + \sum_{k=1}^{\infty} (c_{p,\,p+k}/c_{p,\,p})\,\beta_{p+k} = 0, \quad (p = 0, 1, \ldots). \qquad (31)$$

THEOREM 3. *The system* (29) *possesses positive restrictions if and only if its matrix A is of class R_1.*

PROOF. *Sufficiency.* Let the deleted and rearranged system be

$$\sum_{k=0}^{\infty} a'_{pk}\beta_k = 0 \ (p = 0, 1, 2, \ldots). \tag{32}$$

The principal diagonal minors of $A' = (a'_{pk})$ do not vanish so that by Lemma 3, (32) is equivalent to the triangular system

$$\beta_p + \sum_{k=1}^{\infty} (c'_{p,\, p+k}/c'_{p,\, p})\ \beta_{p+k} = 0 \ (p = 0, 1, 2, \ldots). \tag{33}$$

By Corollary 1.1 there exists a positive restriction of (33). Hence of (32) and hence of (29) inasmuch as the former contains the latter.

Necessity. Given $\chi_n > 0$ such that the only solution of (29) subject to $|\beta_n| \leqslant \chi_n$ is the trivial solution. We wish to show that the matrix A is of class R_1. Firstly, there exists an integer k_0 such that $a_{k_0 0} \neq 0$. For if not, $\beta_0 = \chi_0/2$, $\beta_i = 0$ $(i > 0)$ would be a non-trivial solution with $|\beta_n| \leqslant \chi_n$ $(n = 0, 1, 2, \ldots)$. This contradicts the hypothesis. Secondly, there exists an integer k_1 such that

$$\begin{vmatrix} a_{k_0 0} & a_{k_0 1} \\ a_{k_1 0} & a_{k_1 1} \end{vmatrix} \neq 0.$$

For if not, then for all values of the parameter t, $\beta_0 = t\, a_{k_0 1}$, $\beta_1 = -t\, a_{k_0 0}$, $\beta_i = 0$ $(i > 1)$ will be a solution of (29). By choosing t sufficiently small, we shall obtain a solution with $|\beta_n| \leqslant \chi_n$. It will be a non-trivial solution inasmuch as $a_{k_0 0} \neq 0$. This is impossible. In this way we proceed step by step. The n th step is as follows. Having selected integers $k_0, k_1, \ldots, k_{n-1}$ such that

$$\begin{vmatrix} a_{k_0 0}, & a_{k_0 1}, & \ldots, & a_{k_0 p} \\ \cdot & \cdot & & \cdot \\ a_{k_p 0}, & a_{k_p 1}, & \ldots, & a_{k_p p} \end{vmatrix} \neq 0 \ (p = 0, 1, \ldots, n-1), \tag{34}$$

we consider all the determinants of the form

$$\begin{vmatrix} a_{k_0 0}, \ldots, a_{k_0 n} \\ \cdot \quad \cdot \\ a_{k_{n-1} 0}, \ldots, a_{k_{n-1} n} \\ a_{r0}, \quad \ldots, a_{rn} \end{vmatrix} \qquad 0 \leqslant r < \infty. \qquad (35)$$

They cannot all vanish, for if so, then for all non-zero values of the parameter t, $\beta_0 = t\,\Delta_0$, $\beta_1 = t\,\Delta_1, \ldots, \beta_n = t\,\Delta_n$, $\beta_i = 0$ $(i > n)$, where Δ_i denotes the cofactor of a_{ri} in (35), would constitute a solution of (29). It is a non-trivial solution since by (34), $\Delta_n \neq 0$. By choosing t sufficiently small in absolute value, we may obtain a non-trivial solution with $|\beta_n| \leqslant \chi_n$. This is a contradiction. We may therefore select a sequence of integers $k_0, k_1, \ldots, k_p, \ldots$ such that the determinants (34) do not vanish for all $p \geqslant 0$. If the ith row of a matrix A' is composed of the k_ith row of the matrix A, then A' will have non-vanishing principal diagonal minors. A is thus of class R_1.

THEOREM 4. *The system* (29) *has a semi-positive restriction if and only if its matrix A is of class R_2.*

PROOF. *Sufficiency.* With A of class R_2 we wish to show the existence of a restriction $\{\chi_n^*\}$ with χ_n^* positive for infinitely many n. Designate by n_k' the indices of those columns which when struck from A render it of class R_1, and by n_k the indices of the columns which remain. Consider the residual system

$$\sum_{k=0}^{\infty} a_{p\,n_k}\,\beta_{n_k} = 0 \quad (p = 0, 1, 2, \ldots). \qquad (36)$$

By Theorem 3, there is a positive restriction $\{\chi_{n_k}\}$ for (36). Hence, if we set $\chi_q^* = \chi_q$ $(q = n_k)$, $\chi_q^* = 0$ $(q = n_k')$ we shall have a semi-positive restriction of (29) inasmuch as any solution β_n of (29) subject to $|\beta_n| \leqslant \chi_n^*$ is a solution of (36).

Necessity. Let $\chi_{n_k} > 0$ while $\chi_{n_{k'}} = 0$. Delete the $n_{k'}$th columns from the matrix A and consider the deleted system (36). This system cannot have a non-trivial solution $\beta_{n_k}^*$ with $|\beta_{n_k}^*| \leqslant \chi_{n_k}$, for otherwise, the set

of values $\beta_j = \beta_j^*$ $(j = n_k)$, $\beta_j = 0$ $(j = n_{k'})$ would constitute a non-trivial solution of (29) with $|\beta_j| \leqslant \chi_j$ $(j = 0, 1, 2, \ldots)$. This is impossible by our assumption. Thus (36) has a positive restriction, so that by Theorem 3, its matrix (a_{pn_k}) is of class R_1.

THEOREM 5. *The system* (29) *has a non-zero restriction if and only if its matrix A is of class R_3, i.e. non-zero.*

PROOF. With $A \neq 0$, let $a_{i_0 j_0} \neq 0$. Set $\chi_n = \delta_{n j_0}$ $((\delta_{ij}) = 1)$. Then $\{\chi_n\}$ contains a non-zero term, and $|\beta_n| \leqslant \chi_n$ implies $\beta_n = 0$ for all $n \neq j_0$. The system (29) therefore contains the equation $a_{i_0 j_0} \beta_{j_0} = 0$, so that $\beta_j = 0$ for all j. Conversely, suppose that $\{\chi_n\}$ is a non-zero restriction. Let $\chi_p > 0$. If a_{ij} were zero for all i, j, then $\beta_p = \chi_p/2 < \chi_p$, $\beta_i = 0$ $(i \neq p)$ is a non-trivial solution.

4. **Applications to Interpolation Problems.** In the present section we deal with linear differential operators of infinite order

$$L(d) = \sum_{n=0}^{\infty} a_n d^n \qquad d^n \equiv d^n/dz^n \,|\, z = 0. \quad (37)$$

It is useful [1, 2] to characterize the operator (37) by means of an associated formal power series

$$l(z) = \sum_{n=0}^{\infty} a_n z^n/n!. \quad (38)$$

If a function $g(z)$ is regular in a region which contains the origin in its interior, then we shall say that the operator $L(d)$ (or $l(z)$) is applicable to $g(z)$, if and only if, the series $\sum_{n=0}^{\infty} a_n g^{(n)}(0)$ converges to a value t, and we shall write $L(d)g = [l(z),\ g(z)] = t$. If $g(z)$ has the Maclaurin expansion $g(z) = \sum_{n=0}^{\infty} \beta_n z^n$, then we shall have

$$L(d)g = [l(z), g(z)] = \sum_{n=0}^{\infty} a_n \beta_n. \quad (39)$$

As a short list of special operators, let us note the following

$$[e^{zw}, g(z)] = g(w) \tag{40a}$$

$$[z^n e^{zw}, g(z)] = g^{(n)}(w) \tag{40b}$$

$$[(e^z - 1)^n e^{ew}, g(z)] = \Delta^{(n)} g(w). \tag{40c}$$

The first of the list is the familiar Cauchy-Heaviside shift function e^{zw}.

Given a set of linear operators

$$l_p(z) = \sum_{k=0}^{\infty} a_{pk} z^k / k! \ (p = 0, 1, \ldots)$$

and a set of constants $c_0, c_1, \ldots$, we may pose the following interpolation problem: Does there exist a function $g(z)$ regular in a neighbourhood of the origin and such that

$$[l_p(z), g(z)] = c_p, \quad (p = 0, 1, 2, \ldots)? \tag{41}$$

In view of (39) it will be seen that this problem is equivalent to the solution of the system

$$\sum_{k=0}^{\infty} a_{pk} \beta_k = c_p \quad (p = 0, 1, 2, \ldots) \tag{42}$$

providing the values β_k so obtained are such that the series $\sum_{k=0}^{\infty} \beta_k z^k$ has more than formal meaning. In this way, each result for solutions of systems (42) or of triangular systems (21) may be reformulated as a result for a corresponding interpolation problem. In this section we shall content ourselves with noting only a few of these reformulations.

Theorem 6. *Given a triangular system of linear operators*

$$l_p(z) = \sum_{k=p}^{\infty} a_{pk} z^k / k! \quad (p = 0, 1, 2, \ldots), \tag{43}$$

then there exist two sets of positive constants $\{\chi_p^{(1)}\}$ and $\{\chi_p^{(2)}\}$ such that if $|c_p| \leqslant \chi_p^{(1)}$, the interpolation problem (41) has a solution $g(z) = \sum_{k=0}^{\infty} \beta_k z^k$. The function $g(z)$ is regular in a neighborhood of the origin, and is the sole solution for which $|\beta_k| \leqslant \chi_k^{(2)}$.

Proof. This follows from Corollary 2.1. The regularity of $g(z)$ will follow if (in the proof of Corollary 2.1) we select $d_j^{(2)} = r^{-j}$, $r > 1$.

As examples of triangular sets of operators we may note in particular (40b) with $w = w_p$ ($p = 0, 1, 2, \ldots$) and (40c) with $w = w_p$. In view of the familiar theorem whereby the interpolation problem $g(w_p) = c_p$ ($p = 0, 1, 2, \ldots$) ($w_p \to \infty$) has an entire solution for all sets of values c_p, we may wonder whether a problem (41) has a regular solution for all choices of c_p. That this cannot be the case is easily seen from the problem $g^{(n)}(0) = (n!)^2$ ($n = 0, 1, 2, \ldots$) which has no regular solution. However, Theorem 6 assures us that all triangular problems will possess solutions providing the size of the interpolated values c_p has been restricted. Furthermore, the solution given by (22) is unique in the sense explained above. Reformulating Theorems 3-5 we may state

Theorem 7. *A set of linear operators* $l_p(z) = \sum_{k=0}^{\infty} a_{kp} z^k / k!$ *possesses a positive, semi-positive, or non-zero restriction* $\{\chi_p\}$ *according as the matrix* $A = (a_{pk})$ *is of class* R_1, R_2, *or* R_3.

In particular, let us note the following corollaries.

Corollary 7.1. *Let* a_n *be arbitrary complex numbers. Then there exists a set of positive constants* $\{\chi_n\}$ *such that if* $g(z) = \sum_{n=n}^{\infty} \beta_n z^n$, *then* $|\beta_n| \leqslant \chi_n$ *and* $g^{(n)}(a_n) = 0$ ($n = 0, 1, 2, ..$) *imply* $g(z) \equiv 0$.

Corollary 7.2. *There exist positive constants* $\{\chi_n\}$ *such that* $|\beta_n| \leqslant \chi_n$ *and* $\Delta^{(n)} g(a_n) = 0$ ($n = 0, 1, 2, \ldots$) *imply* $g(z) \equiv 0$.

Corollary 7.3. *Let* a_n *be distinct but otherwise wholly arbitrary complex numbers. Then there exists a positive set* $\{\chi_n\}$ *such that* $|\beta_n| \leqslant \chi_n$ *and* $g(a_n) = 0$ ($n = 0, 1, 2, \ldots$) *imply* $g(z) \equiv 0$.

PROOF. Using (40a) select $l_n(z) = e^{a_n z} = \sum_{j=0}^{\infty} a_n{}^j z^j / j!$. The matrix A is therefore $A = (a_i^j)$, and its principal diagonal determinants are the Vandermonde determinants $\prod_{i<j\leqslant n} (a_i - a_j) \neq 0$. A is therefore of class R_1.

Finally we note that while a triangular set of operators always possesses a positive restriction, a finite set of operators cannot possess even a semi-positive restriction.

In a paper which follows, the method of infinite systems of linear equations will be applied to interpolation problems for entire functions of exponential type.

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EXTENSIONS OF HOMOMORPHISMS AND REGULAR IDEALS

By R. Creighton Buck

The purpose of this note is to obtain a very simple criterion for extending ring homomorphisms, and to show its significance for the L' algebra of an abelian group, and for general rings.

Theorem 1. *Let R be a ring, and let A be an ideal in R, which is contained in the center of R. Let Ω be a ring with unit, and let ϕ be a homomorphism of A onto Ω. Then, ϕ has a unique extension to a homomorphism $\bar{\phi}$ of R onto Ω.*

Choose $u \in A$ with $\phi(u) = 1$, the unit of Ω. Define $\bar{\phi}$ on R by $\bar{\phi}(x) = \phi(xu)$. If $x \in A$, then $\bar{\phi}(x) = \phi(x)$, so that $\bar{\phi}|_A = \phi$. Clearly, $\bar{\phi}(x-y) = \bar{\phi}(x) - \bar{\phi}(y)$. We also have $\bar{\phi}(x)\,\bar{\phi}(y) = \phi(xu)\,\phi(yu) = \phi(xuyu) = \phi(xyu^2) = \phi(xyu) = \bar{\phi}(xy)$. Thus, $\bar{\phi}$ is a homomorphism of R onto Ω. If ψ is any other extension of ϕ, then $\psi(u) = 1$, and for any $x \in R$, $\psi(x) = \psi(xu) = \phi(xu) = \bar{\phi}(x)$.

We also observe that if R has a ring of operators, and ϕ commutes with these, so does $\bar{\phi}$. Likewise, if R and Ω are topological rings, and ϕ is continuous, so is $\bar{\phi}$; in particular, if R and Ω are normed, and ϕ is bounded, so is $\bar{\phi}$.

A special case of this may be of interest. We choose for A, the L' algebra of a locally compact abelian group G, and for R, the algebra generated from A by adjoining all the point functionals θ_b for $b \in G$. These are defined by $\theta_b(f) = f(b)$ for each continuous function f on G. Let ϕ be any continuous homomorphism of A onto the complex field Ω. By the theorem, this has a continuous extension $\bar{\phi}$, a homomorphism of R onto Ω. If we define a function

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χ by $\chi(b) = \bar{\phi}(\theta_b)$, it is easily seen that χ is a continuous character of the group G. (For comparison, see [1]).

We next turn to the extension of ideals in a ring. An ideal I in a ring R is said to be *regular* if R/I is a ring with unit. Let A be any ideal of R, contained in the center.

THEOREM 2. *The correspondence $I^* \to I = I^* \cap A$ effects a one-to-one mapping between the set of all regular ideals in the ring A, and the set of all regular ideals I^* of R having the property that $I^* + A = R$.*

Thus, every regular ideal of A is contained in a unique regular ideal I^* of R such that $I^* + A = R$.

If I^* is a regular ideal of R, and $I^* + A = R$, then $I = I^* \cap A$ is an ideal of A; since R/I^* and A/I are isomorphic, I is regular. Conversely, given any regular ideal I of A, let ϕ be the corresponding homomorphism mapping A onto $\Omega = A/I$. By Theorem 1, we can extend ϕ to a homomorphism $\bar{\phi}$ of R onto Ω. Denoting the kernel of $\bar{\phi}$ by I^*, this is a regular ideal of R. If $x \in R$, then $\bar{\phi}(x) = \bar{\phi}(xu)$, and $x - xu \in I^*$; since $xu \in A$, this shows that $x \in A + I^*$, and $R = I^* + A$. Morevoer, since $\bar{\phi}$ extends ϕ, $I^* \cap A = I$. So far, we have shown that the correspondence $I^* \to I^* \cap A$ maps the set of ideals I^* onto the set of all regular ideals of A. We have only to show that it is one-to-one. Let I_1^* and I_2^* be two regular ideals of R, with $I_1^* \cap A = I = I_2^* \cap A$, and $I_1^* + A = R = I_2 + A$. Since $I_1^*/I \cong R/A \cong I_2^*/I$, we at once have $I_1 = I_2$.

Suppose that R is a commutative ring. We denote the set of all regular maximal ideals of R by $\mathfrak{M}(R)$, and for any proper ideal A in R, let $h(A)$, the hull of A, be the set of all ideals M in $\mathfrak{M}(R)$ with $M \supset A$. (See [2]). It is immediate that there is a natural one-to-one correspondence between $h(A)$ and $\mathfrak{M}(R/A)$. The following result is dual to this.

COROLLARY. *There is a natural one-to-one correspondence between $\mathfrak{M}(A)$ and $\mathfrak{M}(R) - h(A)$, the complement of $h(A)$ in $\mathfrak{M}(R)$.*

We need only observe that a maximal ideal M of R obeys $M+A=R$, if and only if M does not contain A. Both of these natural correspondences are homomorphisms, if the structure spaces $\mathfrak{M}(R)$, $\mathfrak{M}(R/A)$ and $\mathfrak{M}(A)$ are given the Stone topology [3].

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ON THE NORLUND SUMMABILITY OF THE *R*TH DERIVED FOURIER SERIES

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1. Let $f(x)$ be a function of the class $L(0, 2\pi)$ and periodic with period 2π. Let the Fourier series associated with $f(x)$ be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad (1.1)$$

The rth derived series of the Fourier series will be

$$\sum_{n=1}^{\infty} (-1)^{r/2} n^r (a_n \cos nx + b_n \sin nx) \ (r \text{ even}),$$

$$\sum_{n=1}^{\infty} (-1)^{(r-1)/2} n^r (b_n \cos nx - a_n \sin nx) \ (r \text{ odd}). \qquad (1.2)$$

If at a point x the function $f(x)$ satisfies an equation of the form

$$\tfrac{1}{2}[f(x+t)+(-1)^r f(x-t)] = \sum_{i=0}^{[r/2]} f^{(r-2i)}(x) . \frac{t^{r-2i}}{(r-2i)!} + w(x, t) . \frac{t^r}{r!}, \qquad (1.3)$$

where $w(x, t) \to 0$ as $t \to 0$, then $f^{(r)}(x)$ is called the rth generalized derivative of $f(x)$ at the point x after de la Vallée-Poussin [2].

From the existence of the rth generalized derivative follows that of the $(r-2j)$th, $0 < 2j \leqslant r$. Whenever the ordinary rth derivative exists, it is identical with the generalized derivative $f^{(r)}(x)$.

The (C, k) summability of the rth derived series (1.2) has been considered by Zygmund [4] who has proved the following theorem:

A. If the generalized derivative $f^{(r)}(x)$ exists, the rth derived series of $f(x)$ is summable (C, k), $k > r$, to the value $f^{(r)}(x)$.

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Astrachan [1] has generalized this theorem for Norlund summability as follows:

B. A regular Norlund method of summation (N, p) sums the series (1.2) to the generalized derivative $f^{(r)}(x)$ at every point x at which it exists, provided

(i) $n^j\,|\,\Delta^{j-1}p_n\,| = O(\,|\,P_n\,|) \quad (j = 1, 2, \ldots, r+1)$

(ii) $$\sum_{k=1}^{n} k^{j-m-1}(n-k)^m\,|\,\Delta^{j-1}p_k\,| = O(|\,P_n\,|)$$
$$(j = 1, 2, \ldots, r+2\,;\ m = 0, 1, \ldots, r)$$

(iii) $$\sum_{k=1}^{n} \frac{|P_k|}{k^{r+1}} = O\left(\frac{|P_n|}{n^r}\right).$$

Wang [3] has generalized Zygmund's result in another direction by proving the following theorem:

C. The rth derived series of the Fourier series is summable (C, k), $k > r$, to the value $f^{(r)}(x)$ at every point x at which

$$w^*(t) = \int_0^t |\,w(x, u)\,|\,du = o(t).$$

The object of this paper is to give a generalization of Astrachan's theorem analagous to the one given by Wang of Zygmund's theorem. We prove the following theorem:

THEOREM. *The rth derived series of the Fourier series is summable by a regular Norlund method of summation (N, p) to the generalized derivative $f^{(r)}(x)$ at every point x at which*

$$w^*(t) = \int_0^t |\,w(x, u)\,|\,du = o\,(t),$$

provided

(*i*) $n^j\,|\,\Delta^{j-1}p_n\,| = O(\,|\,P_n\,|) \quad (j = 1, 2, \ldots, r+1)$

(*ii*) $$\sum_{k=1}^{n} k^{j-m-1}(n-k)^m\,|\,\Delta^{j-1}p_k\,| = O(\,|\,P_n\,|)$$
$$(j = 1, 2, \ldots, r+2;\ m = 0, 1, \ldots, r)$$

(*iii*) $$\sum_{k=1}^{n} \frac{|P_k|}{k^{r+1}} = O\left(\frac{|P_n|}{n^r}\right).$$

Remark. It should be noted that this condition follows from (*i*) and (*iii*) for $j=1, \ldots, r+1$ and $m=0, 1, \ldots, r$ by virtue of Lemma 1 given below.

2. Let $s_n(x)$ denote the partial sum of the series (1.1) of order n. Then we have

$$s_n(x) = \frac{1}{2\pi}\int_0^{\pi} \{ f(x+t)+f(x-t) \} . \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} dt.$$

If we denote by $N_n [f(x), p_k]$, the nth Norlund mean of the sequence $s_n(x)$ defined by the generating sequence $\{p_n\}$, we have

$$N_n[f(x), p_k] = \int_0^{\pi} \{ f(x+t)+f(x-t) \} N_n(t)\, dt,$$

where the kernel $N_n(t)$ is defined by the relation

$$N_n(t) = \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_{n-k} \frac{\sin (k+\frac{1}{2})t}{\sin \frac{1}{2} t}.$$

If $N_n^{(r)} [f(x), p_k]$ denote the (N, p) mean of the series (1. 2) and $N_n^{(r)}(t)$, the rth derivative of the kernel $N_n(t)$, we have

$$N_n^{(r)}[f(x), p_k] = (-1)^r \int_0^{\pi} \{ f(x+t)+(-1)^r f(x-t) \} N_n^{(r)}(t)\, dt. \tag{2.1}$$

Substituting the value of $f(x+t)+(-1)^r f(x-t)$ given by (1.3) in (2.1), we get

$$N_n^{(r)}[f(x), p_k] = 2(-1)^r \sum_{i=0}^{[\frac{1}{2}r]} f^{(r-2i)}(x). \int_0^{\pi} \frac{t^{r-2i}}{(r-2i)!} N_n^{(r)}(t)\, dt$$
$$+2(-1)^r \int_0^{\pi} w(x, t). \frac{t^r}{r!} N_n^{(r)}(t)\, dt.$$

A simple calculation gives

$$N_n^{(r)}[f(x), p_k] - f^{(r)}(x) = 2(-1)^r \sum_{i=0}^{[\frac{1}{2}r]} U_i (r, n). f^{(r-2i)}(x)$$
$$+2(-1)^r \int_0^{\pi} w(x, t). \frac{t^r}{r!} N_n^{(r)}(t)\, dt,$$

where

$$U_i(r, n) = (-1)^r \sum_{j=i}^{[\frac{1}{2}(r-1)]} \frac{\pi^{2j-2i+1}}{(2j-2i+1)!} N_n^{(2j)}(t) \,|_{t=\pi}.$$

In order to prove the theorem we have to show that

(i) $$\sum_{i=0}^{[\frac{1}{2}r]} U_i(r, n) f^{(r-2i)}(x) = o(1),$$

and

(ii) $$\int_0^\pi w(x, t).\ \ t^r N_n^{(r)}(t)\, dt = o(1), \qquad (2.2)$$

as $n \to \infty$.

Now it is known [1] that under the conditions (i) to (iii) of the theorem $U_i(r, n) \to 0$ as $n \to \infty$. Hence it readily follows that

$$\sum_{i=0}^{[\frac{1}{2}r]} U_i(r, n) f^{(r-2i)}(x) = o(1),$$

as $n \to \infty$.

3. **Lemmas.** In order to prove that the relation (2.2) holds we shall require the following lemmas.

LEMMA 1. *If* $\sum_{k=1}^{n} \frac{|P_k|}{k^{r+1}} = O\left(\frac{|P_n|}{n^r}\right)$,

then

$$\sum_{k=1}^{n} \frac{|P_k|}{k^{m+1}} = O\left(\frac{|P_n|}{n^m}\right) \quad (m = 0, 1, \ldots, r-1).$$

The proof is quite elementary and is therefore omitted.

LEMMA 2. *If* $\sum_{k=1}^{n} k(n-k)^m |\Delta^{m+1} p_k| = O(|P_n|)$

$$(m = 0, 1, \ldots, r)$$

and

$$\sum_{k=1}^{n} \frac{|P_k|}{k^{\beta+1}} = O\left(\frac{|P_n|}{n^\beta}\right) \quad (\beta = 0, 1, \ldots, r),$$

then

$$\sum_{k=0}^{n} (n-k)^m \,|\, \triangle^{m+1} p_k \,| = O\left(\frac{|P_n|}{n} \right) = o\,(P_n) \quad (m = 0, \ldots, r).$$

For proof reference may be made to Astrachan [1].

LEMMA 3. *For* $0 \leqslant t \leqslant n^{-1}$,

$$t^r N_n^{(r)}(t) = O(n).$$

For proof reference may be made to Astrachan [1].

LEMMA 4. *If*

(*i*) $\quad |\triangle^{j-1} p_n| = o\,(|P_n|) \quad (j = 1, 2, \ldots, r+1)$

(*ii*) $\sum_{k=0}^{n} (n-k)^m \,|\, \triangle^{m+1} p_k \,| = o\,(|P_n|) \quad (m = 0, 1, \ldots r)$,

then

$$t^r N_n^{(r)}(t) = o\,(1),$$

uniformly in t *for* $0 < \delta \leqslant |t| \leqslant \pi$, *where* δ *is fixed.*

For proof reference may be made to Astrachan [1].

4. **Proof.**

We shall now proceed to establish (2. 2). We set

$$\int_0^{\pi} w(x, t).\ t^r N_n^{(r)}(t)\,dt$$

$$= \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi} \right) w(x, t).\ t^r N_n^{(r)}(t)\,dt = L_1 + L_2 + L_3.$$

Now

$$L_1 = \int_0^{\frac{1}{n}} w(x,t).\ t^r N_n^{(r)}(t)\,dt$$

$$= O\left(\int_0^{\frac{1}{n}} |\, w(x, t) \,|.\ \,|\, t^r N_n^{(r)}(t) \,|\, dt \right)$$

$$= O\left(n \int_0^{\frac{1}{n}} |\, w(x, t) \,|\, dt \right)$$

$$= O(n.\ w^*(n^{-1}))$$

$$= o(1),$$

by virtue of Lemma 3.

Again

$$L_3 = \int_{\delta}^{\pi} w(x, t).\ t^r N_n^{(r)}(t)\,dt$$

$$O\left(\int_{\delta}^{\pi}|w(x,t)|.\ |t^r N_n^{(r)}(t)|\,dt\right)$$
$$=o\left(\int_{\delta}^{\pi}|w(x,t)|\,dt\right)$$
$$=o(1),$$

by virtue of Lemma 4.

Lastly we have to show that

$$L_2=\int_{\frac{1}{n}}^{\delta} w(x,t).\ t^r N_n^{(r)}(t)dt=o\ (t).$$

We require a suitable estimate for $t^r N_n^{(r)}(t)$ in (n^{-1},δ). Such an estimate has been given by Astrachan [1]. In fact,

$$|t^r N_n^{(r)}(t)|\leqslant A\sum_{m=1}^{4} M_{nm}(t),$$

where

$$M_{n1}(t)=\frac{1}{R(n)}\sum_{\beta=0}^{r} n^{\beta}\, t^{\beta-1}R(t^{-1}),$$

$$M_{n2}(t)=\frac{1}{R(n)}\sum_{j=1}^{r+1}\sum_{\beta=0}^{r}(n-\tau)^{\beta}\, t^{\beta-j-1}\,|\Delta^{j-1}p_{\tau}|,$$

$$M_{n3}(t)=\frac{1}{R(n)}\sum_{j=1}^{r+1} t^{-j-1}\,|\Delta^{j-1}p_n|,$$

$$M_{n4}(t)=\frac{1}{R(n)}\sum_{\beta=0}^{r} t^{\beta-r-2}\left\{W^{\beta}(n)-W^{\beta}(t^{-1})\right\},$$

with the following notations :

$$r_n=|p_n|,\qquad R_n=r_0+r_1+\ldots+r_n;$$
$$r(u)=r_{[u]},\qquad R(u)=R_{[u]};$$
$$W_m^{\beta}=\sum_{k=0}^{m}(n-k)^{\beta}\,|\Delta^{r+1}p_k|,\ W^{\beta}(u)=W_{[u]}^{\beta},\ \tau=[t^{-1}];$$

$[u]$ denoting the greatest integer $\leqslant u$.

We, therefore, have

$$L_2=\int_{\frac{1}{n}}^{\delta} w(x,t).\ t^r N_n^{(r)}(t)dt$$
$$=O\left(\int_{\frac{1}{n}}^{\delta}|w(x,t)|.\ |t^r N_n^{(r)}(t)|\,dt\right)$$

$$= O\left\{ w^*(t) \,|\, t^r N_n^{(r)}(t) \,|\right\} \Big|_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{\delta} w^*(t).\ d\,|\, t^r N_n^{(r)}(t) \,|$$

since, by virtue of Lemma 4,

$$w^*(\delta) \,|\, \delta^r N_n^{(r)}(\delta) \,| = O(\,|\, \delta^r N_n^{(r)}(\delta) \,|\,) = o\,(1),$$

and, by virtue of Lemma 3 and condition $w^*(t) = o\,(t)$,

$$w^*(n^{-1}) \,|\, n^{-r} N_n^{(r)}(n^{-1}) \,| = O\,(w^*(n^{-1}).\ n) = o\,(1),$$

we have only to show that

$$\int_{\frac{1}{n}}^{\delta} w^*(t).\ \left\{ d\, M_{nm}(t) \right\} = o\,(1) \qquad (m = 1, 2, 3, 4).$$

For $m = 1$,

$$\int_{\frac{1}{n}}^{\delta} w^*(t)\ \left\{ d\, M_{nm}(t) \right\} = \int_{\frac{1}{n}}^{\delta} w^*(t) \frac{1}{R(n)} \sum_{\beta=0}^{r} n^{\beta}\, t^{\beta-1} dR(t^{-1})$$

$$+ \int_{\frac{1}{n}}^{\delta} w^*(t). \frac{1}{R(n)} \sum_{\beta=0}^{r} (\beta - 1)\ n^{\beta}\ t^{\beta-2} R(t^{-1}) dt$$

$$= o\left(\int_1^n \frac{1}{R(n)} \sum_{\beta=0}^{r} n^{\beta} s^{-\beta} \,|\, dR(s) \,|\, \right)$$

$$+ o\left(\int_1^n \frac{1}{R(n)} \sum_{\beta=0}^{r} n^{\beta} s^{-\beta-1} R(s) ds \right)$$

$$= o\left(\sum_{\beta=0}^{r} \frac{n^{\beta}}{|P_n|} \sum_{k=1}^{r} \frac{|p_k|}{k^{\beta}} \right) + o\left(\sum_{\beta=0}^{r} \frac{n^{\beta}}{|P_n|} \sum_{k=1}^{n} \frac{|P_k|}{k^{\beta+1}} \right).$$

Now

$$\sum_{\beta=0}^{r} \frac{n^{\beta}}{|P_n|} \sum_{k=1}^{n} \frac{|p_k|}{k^{\beta}} = O\left(\sum_{\beta=0}^{r} \frac{n^{\beta}}{|P_n|} \sum_{k=1}^{n} \frac{|P_k|}{k^{\beta+1}} \right)$$

$$= O\,(1),$$

by virtue of conditions (*i*) and (*iii*) and Lemma 1;

$\displaystyle \sum_{\beta=0}^{r} \frac{n^{\beta}}{|P_n|} \sum_{k=1}^{r} \frac{|P_k|}{k^{\beta+1}} = O(1)$, by virtue of condition (*iii*) and Lemma 1. Hence

$$\int_{\frac{1}{n}}^{\delta} w^*(t)\ \left\{ dM_{n1}(t) \right\} = o(1).$$

For $m=2$,

$$\int_{\frac{1}{n}}^{\delta} w^*(t) \left\{ dM_{n2}(t) \right\}$$

$$= \int_{\frac{1}{n}}^{\delta} w^*(t) \cdot \frac{1}{R(n)} \sum_{j=1}^{r+1} \sum_{\beta=0}^{r} (n-\tau)^{\beta} t^{\beta-j-1} d_t \left| \Delta^{j-1} p_\tau \right|$$

$$- \int_{\frac{1}{n}}^{\delta} w^*(t) \cdot \frac{1}{R(n)} \sum_{j=1}^{r+1} \sum_{\beta=0}^{r} \beta (n-\tau)^{\beta-1} t^{\beta-j-1} \left| \Delta^{j-1} p_\tau \right| d_t \tau$$

$$+ \int_{\frac{1}{n}}^{\delta} w^*(t) \cdot \frac{1}{R(n)} \sum_{j=1}^{r+1} \sum_{\beta=0}^{r} (\beta-j-1)\ (n-\tau)^{\beta} t^{\beta-j-2} \left| \Delta^{j-1} p_\tau \right| dt$$

$$= o\left(\int_1^n \frac{1}{R(n)} \sum_{j=1}^{r+1} \sum_{\beta=0}^{r} (n-[s])^{\beta} s^{j-\beta} d \left| \Delta^{j-1} p_{[s]} \right| \right)$$

$$+ o\left(\int_1^n \frac{1}{R(n)} \sum_{j=1}^{r+1} \sum_{\beta=1}^{r} (n-[s])^{\beta-1} s^{j-\beta} \left| \Delta^{j-1} p_{[s]} \right| d[s] \right)$$

$$+ o\left(\int_1^n \frac{1}{R(n)} \sum_{j=1}^{r+1} \sum_{\beta=0}^{r} (n-[s])^{\beta} s^{j-\beta-1} \left| \Delta^{j-1} p_{[s]} \right| ds \right).$$

$$= o\left(\sum_{j=1}^{r+1} \sum_{\beta=0}^{r} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{j-\beta} (n-k)^{\beta} \left| \Delta^{j} p_k \right| \right)$$

$$+ o\left(\sum_{j=1}^{r+1} \sum_{\beta=1}^{r} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{j-\beta} (n-k)^{\beta-1} \left| \Delta^{j-1} p_k \right| \right)$$

$$+ o\left(\sum_{j=1}^{r+1} \sum_{\beta=0}^{r} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{j-\beta-1} (n-k)^{\beta} \left| \Delta^{j-1} p_k \right| \right).$$

Now

$$\sum_{j=1}^{r+1} \sum_{\beta=0}^{r} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{j-\beta} (n-k)^{\beta} \left| \Delta^{j} p_k \right| = O(1),$$

since

$$\sum_{k=1}^{n} k^{j-\beta} (n-k)^{\beta} \left| \Delta^{j} p_k \right| = O(|P_n|)$$

$$(j=1, 2, \ldots, r+1;\ \beta=0, 1, \ldots, r)$$

by virtue of condition (*ii*) of the theorem;

and

$$\sum_{j=1}^{r+1}\sum_{\beta=1}^{r}\frac{1}{|P_n|}\sum_{k=1}^{n}k^{j-\beta}(n-k)^{\beta-1}|\Delta^{j-1}p_k| = O(1),$$

since

$$\sum_{k=1}^{n}k^{j-\beta}(n-k)^{\beta-1}|\Delta^{j-1}p_k| = O(|P_n|)$$

$$(j=1, 2, .., r+1;\ \beta=1, 2, .., r)$$

by virtue of condition (*ii*) of the theorem;

and lastly

$$\sum_{j=1}^{r+1}\sum_{\beta=0}^{r}\frac{1}{|P_n|}\sum_{k=1}^{n}k^{j-\beta-1}(n-k)^{\beta}|\Delta^{j-1}p_k| = O(1),$$

since

$$\sum_{k=1}^{n}k^{j-\beta-1}(n-k)^{\beta}|\Delta^{j-1}p_k| = O(|P_n|)$$

$$(j=1, 2, \ldots, r+1;\ \beta=0, 1, \ldots, r)$$

by virtue of condition (*ii*) of the theorem.

Hence

$$\int_{\frac{1}{n}}^{\delta} w^*(t)\ \{dM_{n2}(t)\} = o(1).$$

For $m=3$,

$$\int_{\frac{1}{n}}^{\delta} w^*(t)\ \{dM_{n3}(t)\}$$

$$= -\int_{\frac{1}{n}}^{\delta} w^*(t).\ \frac{1}{R(n)}\sum_{j=1}^{r+1}(j+1)t^{-j-2}|\Delta^{j-1}p_n|\,dt$$

$$= o\left(\int_{\frac{1}{n}}^{\delta}\frac{1}{R(n)}\sum_{j=1}^{r+1}t^{-j-1}|\Delta^{j-1}p_n|\,dt\right)$$

$$= o\left(\frac{1}{|P_n|}\sum_{j=1}^{r+1}n^{j}|\Delta^{j-1}p_n|\right)$$

since

$$\int_{\frac{1}{n}}^{\delta} \frac{1}{R(n)} \sum_{j=1}^{r+1} t^{-j-1} |\Delta^{j-1} p_n| \, dt$$

$$= \frac{1}{R(n)} \sum_{j=1}^{r+1} |\Delta^{j-1} p_n| \int_{\frac{1}{n}}^{\delta} t^{-j-1} dt$$

$$= \frac{1}{R(n)} \sum_{j=1}^{r+1} |\Delta^{j-1} p_n| \left[-\frac{t^{-j}}{j} \right]_{\frac{1}{n}}^{\delta}$$

$$= O\left(\frac{1}{R(n)} \sum_{j=1}^{r+1} n^j |\Delta^{j-1} p_n| \right)$$

$$= O\left(\frac{1}{|P_n|} \sum_{j=1}^{r+1} n^j |\Delta^{j-1} p_n| \right).$$

Now

$$\sum_{j=1}^{r+1} |\Delta^{j-1} p_n| = o\,(|P_n|)$$

since

$$|\Delta^{j-1} p_n| = o\,(|P_n|) \quad (j = 1, 2, \ldots, r+1)$$

by virtue of condition (i);

and

$$\sum_{j=1}^{r+1} n^j |\Delta^{j-1} p_n| = O\,(|P_n|)$$

since condition (i) holds.

Hence

$$\int_{\frac{1}{n}}^{\delta} w^*(t). \, \{ dM_{n3}(t) \} = o(1).$$

For $m = 4$,

$$\int_{\frac{1}{n}}^{\delta} w^*(t) \, \{ dM_{n4}(t) \}$$

$$= -\int_{\frac{1}{n}}^{\delta} w^*(t) \, \frac{1}{R(n)} \sum_{\beta=0}^{r} t^{\beta-r-2} dW^{\beta}(t^{-1})$$

$$+ \int_{\frac{1}{n}}^{\delta} w^*(t) \, \frac{1}{R(n)} \sum_{\beta=0}^{r} (\beta-r-2) t^{\beta-r-3} \{ W^{\beta}(n) - W^{\beta}(t^{-1}) \} dt$$

$$= o\left(\int_1^n \frac{1}{R(n)} \sum_{\beta=0}^{r} s^{r-\beta+1} dW^{\beta}(s)\right)$$

$$+ o\left(\int_1^n \frac{1}{R(n)} \sum_{\beta=0}^{r} s^{r-\beta} \{ W^{\beta}(n) - W^{\beta}(s) \} ds\right).$$

Now

$$\int_1^n \frac{1}{R(n)} \sum_{\beta=0}^{r} s^{r-\beta} \{ W^{\beta}(n) - W^{\beta}(s) \} ds$$

$$= \left[\frac{1}{R(n)} \sum_{\beta=0}^{r} \frac{s^{r-\beta+1}}{r-\beta+1} \{ W^{\beta}(n) - W^{\beta}(s) \}\right]_1^n$$

$$+ \int_1^n \frac{1}{R(n)} \sum_{\beta=0}^{r} s^{r-\beta+1} dW^{\beta}(s).$$

$$= O\left(\frac{1}{R(n)} \sum_{\beta=0}^{r} W^{\beta}(n)\right) + \int_1^n \frac{1}{R(n)} \sum_{\beta=0}^{r} s^{r-\beta+1} dW^{\beta}(s).$$

Hence

$$\int_{\frac{1}{n}}^{\delta} w^*(t) \{ dM_{n4}(t) \}$$

$$= O\left(\frac{1}{R(n)} \sum_{\beta=0}^{r} W^{\beta}(n)\right) + o\left(\int_1^n \frac{1}{R(n)} \sum_{\beta=0}^{r} s^{r-\beta+1} dW^{\beta}(s)\right)$$

$$= O\left(\sum_{\beta=0}^{r} \frac{1}{|P_n|} \sum_{k=0}^{n} (n-k)^{\beta} |\Delta^{r+1} p_k|\right)$$

$$+ o\left(\sum_{\beta=0}^{r} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{r-\beta+1} (n-k)^{\beta} |\Delta^{r+1} p_k|\right).$$

Now

$$\sum_{\beta=0}^{r} \frac{1}{|P_n|} \sum_{k=0}^{n} (n-k)^{\beta} |\Delta^{r+1} p_k| = o\,(1),$$

since

$$\sum_{k=0}^{n} (n-k)^{\beta} |\Delta^{r+1} p_k| = o\,(|P_n|) \quad (\beta = 0, 1, \ldots, r)$$

by virtue of Lemma 2 ;

and

$$\sum_{\beta=0} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{r-\beta+1} (n-k)^{\beta} |\Delta^{r+1} p_k| = O(1).$$

by virtue of condition (ii) of the theorem.

Hence

$$\int_{\frac{1}{n}}^{\delta} u^*(t) \left\{ dM_{n4}(t) \right\} = o(1).$$

Combining these results, we get

$$L_2 = o(1).$$

This completes the proof of the theorem.

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RATIO OF THE DISTANCE TO THE CENTRAL POINT AND THE PARAMETER OF DISTRIBUTION IN A LINE CONGRUENCE

By R S Mishra

1. Let the coordinates of a point on the surface of reference be given by x^i $(i = 1, 2, 3)$ and the direction cosines of the ray 'l' of the congruence be given by λ^i $(i = 1, 2, 3)$. Then the two quadratic forms used by Kummer [3] are

$$f = G_{\alpha\beta}\, du^\alpha\, du^\beta \text{ and } \varphi = \mu_{\alpha\beta}\, du^\alpha\, du^\beta \quad (\alpha, \beta = 1, 2) \qquad (1.1)$$

where
$$G_{\alpha\beta} = \lambda^i_{,\alpha} \cdot \lambda^i_{,\beta} \text{ and } \mu_{\alpha\beta} = \lambda^i_{,\alpha} \cdot x^i_{,\beta}. \qquad (1.2)$$

$\lambda^i_{,\alpha}$ and $x^i_{,\alpha}$ denoting the covariant differentiation of λ^i and x^i respectively with regard to u^α. If

$$E_{\alpha\beta} = (\lambda^i_{,\alpha}\ \ \lambda^i_{,\beta}\ \ \lambda^i) \qquad (1.3)$$

and $E^{\alpha\beta}$ is a tensor conjugate to $E_{\alpha\beta}$, then the two quadratic forms used by Sannia [2, p. 459] are

$$f = G_{\alpha\beta}\, du^\alpha\, du^\beta \text{ and } \theta = \xi_{\alpha\beta}\, du^\alpha\, du^\beta, \qquad (1.4)$$

where
$$\xi_{\alpha\beta} = E^{\gamma\delta}\, G_{\alpha\gamma}\, \mu_{\delta\beta} \text{ or } \mu_{\delta\beta} = \xi_{\alpha\beta}\, E_{\gamma\delta}\, G^{\alpha\gamma}. \qquad (1.5)$$

Then the ratio of the distance to the central point from the surface of reference and the parameter of distribution for any surface through the ray is given by [1, pp. 51, 56]

$$p = -\frac{\mu_{\alpha\beta}\, du^\alpha\, du^\beta}{\xi_{\alpha\beta}\, du^\alpha\, du^\beta} \qquad (1.6)$$

or
$$p = \frac{\xi_{\delta\beta}\, E_{\alpha\gamma}\, G^{\delta\gamma}\, du^\alpha\, du^\beta}{\xi_{\beta\alpha}\, du^\alpha\, du^\beta} \qquad (1.7)$$

by virtue of (1.5)

In (1.7) 'p' varies wrth the value of $\frac{du^1}{du^2}$, ie. with the ruled surface through 'l'. The values of 'p' have finite maximum and minimum. The maximum and minimum values of 'p' are given by

$$E^{\alpha\gamma} E^{\beta\delta} (\xi_{\alpha\beta}+\xi_{\beta\alpha}) (\xi_{\delta\gamma}+\xi_{\gamma\delta})\, p^2 + 2 E^{\alpha\gamma} E^{\beta\delta} (\xi_{\alpha\beta}+\xi_{\beta\alpha}) (\mu_{\delta\gamma}+\mu_{\gamma\delta})\, p + E^{\alpha\gamma} E^{\beta\delta} (\mu_{\alpha\beta}+\mu_{\beta\alpha}) (\mu_{\delta\gamma}+\mu_{\gamma\delta}) = 0. \qquad (1.8)$$

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If 'p_1' and 'p_2' be two values of 'p' in this equation, let us call them 'Principal Ratios'. We shall define the mean ratio and the total ratio by (p_1+p_2) and $(p_1 p_2)$ respectively.

Hence

$$p_1+p_2=-\frac{2E^{\alpha\gamma}E^{\beta\delta}(\xi_{\alpha\beta}+\xi_{\beta\alpha})(\mu_{\delta\gamma}+\mu_{\gamma\delta})}{E^{\alpha\gamma}E^{\beta\delta}(\xi_{\alpha\beta}+\xi_{\beta\alpha})(\xi_{\delta\gamma}+\xi_{\gamma\delta})}$$

which with the help of (1.5) becomes

$$p_1+p_2=\frac{\xi_{\gamma\delta}\,\mu_{\alpha\beta}\,G^{\alpha\beta}\,G^{\gamma\delta}}{\frac{1}{8}E^{\alpha\beta}E^{\gamma\delta}(\xi_{\alpha\gamma}+\xi_{\gamma\alpha})(\xi_{\beta\delta}+\xi_{\delta\beta})}. \qquad (1.9)$$

If d_1 and d_2 are principal parameters and t_1 and t_2 are distances to the central point from the surface of reference, then [1, pp 52, 57]

$$d_1+d_2=\xi_{\gamma\delta}\,G^{\gamma\delta};\quad t_1+t_2=-\mu_{\alpha\beta}\,G^{\alpha\beta}$$

and $$K\equiv d_1 d_2=\tfrac{1}{8}E^{\alpha\beta}E^{\gamma\delta}(\xi_{\alpha\gamma}+\xi_{\gamma\alpha})(\xi_{\beta\delta}+\xi_{\delta\beta}).$$

The equation (1 9) assumes the form

$$p_1+p_2=-\frac{1}{K}(t_1+t_2)(d_1+d_2). \qquad (1.10)$$

From this equation we see that *the mean ratio vanishes when either (a) the congruence is normal or (b) the surface of reference is the middle surface.*

2. In order to find the surfaces for which 'p' has these limiting values the equation (1.7) is differentiated with respect to $t\left(=\frac{du^1}{du^2}\right)$ and $\frac{dp}{dt}$ is put equal to o. Hence these surfaces are

$$\begin{vmatrix}(\mu_{1\alpha}+\mu_{\alpha 1})du^\alpha & (\mu_{2\alpha}+\mu_{\alpha 2})du^\alpha\\(\xi_{1\beta}+\xi_{\beta 1})du^\beta & (\xi_{2\beta}+\xi_{\beta 2})du^\beta\end{vmatrix}=0$$

or $$E^{\gamma\delta}(\mu_{\gamma\alpha}+\mu_{\alpha\gamma})(\xi_{\beta\delta}+\xi_{\delta\beta})du^\alpha du^\beta=0. \qquad (2.1)$$

We know that the equation of the characteristic surfaces is given by [5]

$$J(\theta, J(f,\theta))=0.$$

If the surface of reference is the middle surface $J(f,\theta)=0$ is the same as $\mu_{\alpha\beta}\,du^\alpha du^\beta=0$,

and $$J(\theta, J(f,\theta))\equiv J(\xi_{\alpha\beta}\,du^\alpha du^\beta,\ \mu_{\alpha\beta}\,du^\alpha du^\beta)=0 \qquad (2.2)$$

or $$E^{\gamma\delta}\,(\mu_{\gamma\alpha}+\mu_{\alpha\gamma})\,\xi_{\delta\beta}\,du^\alpha\,du^\beta = 0 \qquad (2.3)$$
since $$E^{\alpha\beta}\,\xi_{\alpha\beta} = 0.$$

The equations (2.1) and (2.3) are the same if $E^{\alpha\beta}\,\xi_{\alpha\beta} = 0$, i.e. if the surface of reference is the middle surface.

HENCE *the surfaces for which the ratio between the distances to the central point and the parameter of distribution is extremum coincide with characteristic surfaces provided the surface of reference is the middle surface.*

3. Characteristic surfaces are one of the five families of surfaces obtained by the repeated application of operation of forming the Jacobian from Sannia's quadratic forms and are given by $J(\theta, J\,(f, \theta)) = 0$.

Let us take the surfaces $J(\varphi, J(f, \varphi)) = 0$, which are one of the five families of surfaces obtained by the repeated application of the operation of forming the Jacobian from Kummer's quadratic forms [4].

For a normal congruence $J(f, \varphi) = 0$ is the same as $\xi_{\alpha\beta}\,du^\alpha\,du^\beta = 0$, so

$$J(\varphi, J(f, \varphi)) \equiv J(\mu_{\alpha\beta}\,du^\alpha\,du^\beta,\ \xi_{\alpha\beta}\,du^\alpha\,du^\beta) = 0 \qquad (3.2)$$
or $$E^{\gamma\delta}\,(\xi_{\beta\delta}+\xi_{\delta\beta})\mu_{\alpha\gamma}\,du^\alpha\,du^\beta = 0 \qquad (3.3)$$
since $$E^{\alpha\beta}\,\mu_{\alpha\beta} = 0.$$

The equations (3.3) and (2.1) are the same if $E^{\alpha\beta}\,\mu_{\alpha\beta} = 0$, i.e. if the congruence is normal. Hence

In a normal rectilinear congruence the surfaces for which the ratio between the distances to the central point and the parameter of distribution is extremum coincide with the surfaces $J(\varphi, J(f, \varphi)) = 0$.

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